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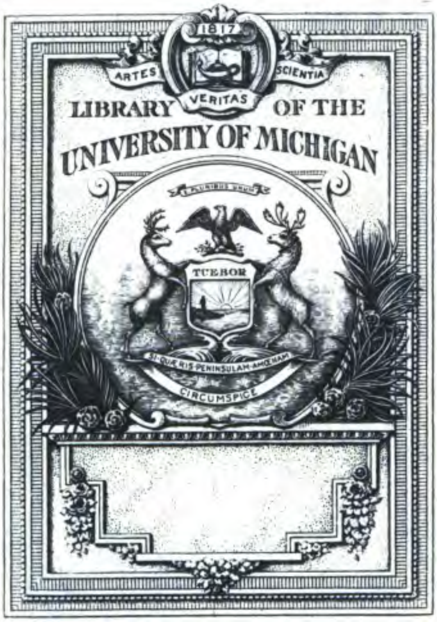
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ELEMENTS

OF THE

CONIC SECTIONS,

By the late Dr ROBERT SIMSON, Profeſſor of
Mathematics in the Univerſity of Glaſgow.

THE FIRST THREE BOOKS,

Tranſlated from the LATIN original.

For the uſe of Students of Mathematics.

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The firſt three books of Dr Simſon's treatiſe of the CONIC SECTIONS are tranſlated into Engliſh, with a view to facilitate the ſtudy of the higher Geometry. Theſe books contain as much of the doctrine as uſually enters into an academical education: but if this ſpecimen ſhall be found uſeful, and be honoured with the approbation of the Learned, the Public will be preſented with a tranſlation of the whole. A work compoſed by a Briton, and which is uſeful over all Europe, would ſeem to be a proper and valuable addition to the ſtock of Engliſh Literature.

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ELEMENTS

OF THE

CONIC SECTIONS.

BOOK I.

Of the PARABOLA.

DEFINITIONS.

I. A Straight line AB, and C a point without it, are given in poſition. On the plane of AB, C there is placed a ruler DEF, having its ſide DE applied to AB, and its other ſide EF on the ſame ſide of AB with the point C. A ſtring FGC is taken equal in length to EF: and one end of this ſtring being fixed in F, and the other in C, a part of Fig. 1.

it FG is, by means of a pin G, brought cloſe to the ſide FE of the ruler; then, the ſtring being kept uniformly tenſe by the pin, the ſide DE of the ruler is moved along AB: And thus the point of the pin, as it moves onwards with the ruler, deſcribes upon the plane a line, named the PARABOLA. This line may be extended to a diſtance from the point C, exceeding any given diſtance, provided the length of the ſide FE of the ruler employed, be greater than that given diſtance.

II. The ſtraight line AB is named the *directrix*.

III. And the point C is named the *focus* of the parabola.

IV. A ſtraight line perpendicular to the directrix, is named a *diameter*; and the point where a diameter meets the parabola, is named the *vertex of that diameter*; and the diameter which paſſes through the focus, is named the *axis of the parabola*; and the vertex of the axis is named the *principal vertex*.

V. When a ſtraight line terminated both ways by a parabola, is biſected by a diameter, it is ſaid to be *ordinately applied* to that diameter; or, it is named, ſimply, an *ordinate* to that diameter.

VI. A ſtraight line quadruple of that ſegment of a diameter which is intercepted between its vertex and the directrix, is named the *latus rectum*, or the *parameter*, of that diameter.

VII. A ſtraight line meeting a parabola only in one point, and which, when produced both ways, falls without the parabola, is ſaid to *touch* the parabola in that point.

PROPOSITION I. THEOREM.

A ſtraight line drawn perpendicular to the directrix from any point of a parabola, is equal to the ſtraight line drawn to the focus from that ſame point.

Let G be a point in the parabola, and GE a ſtraight line perpendicular to the directrix AB; draw GC to the focus C, and let EF be equal to the length of that ſide of the ruler which is on the ſame ſide of AB with the focus C: therefore EF is equal to the length of the ſtring FGC: Fig. 1.

FGC: take away the common part FG, and the remainder EG will be equal to the remainder GC.

Corollary. Hence that ſegment of the axis which is intercepted between the focus and the directrix, is biſected in the vertex of the axis. Thus CB is biſected in H.

PROP. II. Theor.

If the diſtance of any point from the focus of a parabola be equal to the perpendicular drawn from that point to the directrix, that ſame point is in the parabola.

Fig. 1. n. 2. Let there be a parabola, the directrix of which is AB, and the focus C; and let D be a point, the diſtance of which from the focus is the ſtraight line DC; from D draw DE perpendicular to the directrix. If DC be equal to DE, the point D is in the parabola.

From the center C, at the diſtance CD, deſcribe a circle, meeting the axis in the point F: let H be the vertex of the axis, and join CE: then, becauſe any two ſides of a triangle are together greater [20. 1. Elements of Euclid] than the third ſide, CD, DE are together greater than CE; much more, then, are they together [19. 1. Elem.] greater than CB: but CD is equal to DE, as alſo CH [cor. 1. 1.] to HB: therefore CD, that is, CF, is greater than CH: the parabola, therefore, with reſpect to its vertex H, is within the circle GDF: of conſequence, it muſt meet the circle ſome where, ſince it may be extended [def. 1.] to a diſtance from the focus C which ſhall exceed any given diſtance. Now it meets the circle in the point D; for if this is not true, it muſt meet the circle in ſome other point. Let, then, the point L, which is on the ſame ſide of the axis with the point D, be that other point; then, having joined CL, draw LM perpendicular to the directrix, and LN parallel to the ſame; and let LN meet DE in N; and, becauſe the point L is in the parabola, CL is [1. 1.] equal to LM; and, according to the hypotheſis, CD is equal to DE; and, C being the center of the circle, CL is equal to CD: therefore LM, that is, NE, is equal to DE; which is impoſſible: the parabola, therefore,

fore, meets not the circle in the point L, nor any where but in D: therefore D is a point in the parabola.

PROP. III. THEOR.

Any ſtraight line drawn through the focus meets the parabola; and a ſtraight line drawn from any point within a parabola to the focus, is leſs than the perpendicular drawn from that point to the directrix. A ſtraight line, on the other hand, drawn from any point without a parabola to the focus, is greater than the perpendicular drawn from that point to the directrix.

Fig. 1. Let there be a parabola, the directrix of which is AB, and the point C the focus; any ſtraight line drawn through C meets that parabola.

Firſt, if CB, a ſtraight line drawn through the focus, be perpendicular to the directrix, the point H, biſecting [cor. 1. 1.] the ſegment, intercepted between the focus and the directrix, is in the [2. 1.] parabola: but if any other ſtraight line CP be drawn through the focus, biſect the angle BCP by the ſtraight line CM, and let CM meet the directrix in M, and draw MN parallel to the axis BC: then, becauſe the angles PCM, CMN are together leſs than two right angles, for each of them is leſs than one right angle, the ſtraight lines CP, MN meet each other; let them meet in the point O, then the angle OCM is equal to the angle CMO; for each of the two is equal to [29. 1. Elem.] BCM; of conſequence OM is [6. 1. Elem.] equal to OC: therefore the point O is in [2. 1.] the parabola.

To proceed to demonſtrate the other part of the propoſition: Firſt, let there be any point K within the parabola, that is, let it be on the ſame ſide of it with the focus C, and draw KL at right angles to the directrix; draw likewiſe KC to the focus; KC is leſs than KL. Let CK meet the parabola in O, and let there be drawn to the directrix the ſtraight line OM parallel to KL, and let OL be joined. Since, then, the point O is in the parabola, OC is equal to OM; but OM is [19. 1. Elem.] leſs than OL; much more therefore is (OM, or) OC leſs than [20. 1. Elem.] OK and KL together: take away the common part OK, and the remainder KC is leſs than the remainder KL.

Next, let the point Q be without the parabola, and draw QR at right angles to the directrix: QC drawn to the focus is greater than QR. Let QC meet the parabola in O, and draw OM parallel to QR, and join QM: therefore, becauſe CO is equal to OM, CQ is equal to MO together with OQ; but MO together with OQ, is greater than QM; much more, then, are they greater than QR. QC is, therefore, greater than QR.

Cor. Hence it is evident, that any point is within or without a parabola, according as the diſtance of that point from the focus is leſs or greater than a perpendicular drawn from that ſame point to the directrix.

PROP.

PROP. IV. THEOR.

A perpendicular to the directrix meets the parabola only in one point; and when produced downwards, it falls within the parabola.

Let MT be perpendicular to the directrix AB; draw MC to the focus; let CO be drawn, making the angle MCO equal to CMO, and meeting MT in O; OM is conſequently equal to OC; and therefore the point O is in [2. 1.] the parabola. Fig. 1.

Next, take any point T in MO produced, and join TC: ſince, then, the angle MCT is greater than MCO, that is, than the angle CMT, TM is greater than TC: the point T therefore is within [cor. 3. 1.] the parabola. In like manner it may be demonſtrated, that any point above O in the ſtraight line MT is without the parabola.

PROP. V. THEOR.

If from a point in a parabola a ſtraight line be drawn to the focus, and if from the ſame point a ſtraight line be drawn perpendicular to the directrix; the ſtraight line which biſects the angle contained by theſe two ſtraight lines, touches the parabola in that point. Alſo a ſtraight line drawn through the vertex of the axis at right angles to the axis, touches the parabola.

Fig. 2. 1. D being a point in a parabola, and DC drawn from D to the focus, and DA drawn perpendicular to the directrix AB; DE, that biſects the angle CDA, touches the parabola in the point D.

In DE take any other point F; and having joined FA, FC, AC, draw FG perpendicular to the directrix: then, becauſe DA is equal [1. 1. Elem.] to DC, DF com-

mon

mon, and the angle FDA equal to FDC; FC is equal [4. 1. Elem.] to FA; and, conſequently, greater than FG: therefore the point F is without the [cor. 3. 1.] parabola; and, conſequently, the ſtraight line DE touches the parabola [def. 7.] in the point D.

2. HK, a ſtraight line drawn through the vertex of the axis, and made perpendicular to the axis, touches the parabola. In HK take any point K; from which draw KL perpendicular to the directrix; and join KC: and, becauſe KC is greater [19. 1. Elem.] than CH, that is, than [cor. 1. 1.] HB, that is, than KL, KC is greater than KL: therefore the point K is without the parabola, and HK touches the parabola. Fig. 2.

COR. 1. This propoſition points out a method of drawing a ſtraight line that will touch a parabola in a given point, provided the directrix and the focus be given in poſition.

COR. 2. And ſince it has been proved, that all ſtraight lines that touch a parabola, fall without it towards the ſame parts, that curve, it is plain, is every where convex on the ſide on which the touching lines are, but concave on the contrary ſide.

PROP. VI. PROBLEM.

The directrix and the focus of a parabola, and a ſtraight line not parallel to any diameter, being given in poſition; to draw a ſtraight line parallel to the ſtraight line not parallel to any diameter, and which will touch the parabola.

Fig. 2. AB being the directrix and C the focus of a parabola, and MN a ſtraight line not parallel to any diameter; it is required to draw a ſtraight line parallel to MN, and which will touch the parabola.

From the focus C draw CO perpendicular to MN, and meeting the directrix A; and having biſected AC in E, draw ED parallel to MN, and meeting the diameter through A in the point D, and join CD;

then, in the triangles ADE, CDE, AE is equal to CE, ED common, and the angles at E right angles; DA, therefore, is equal to DC; and, conſequently, the point D is in the [2. 1.] parabola: And, ſince the angle ADE is equal to the angle CDE, the ſtraight line DE, as was ſhewn in the preceding propoſition, touches the parabola.

PROP. VII. THEOR.

If from a point in a parabola, there be drawn a ſtraight line, neither parallel to the axis, nor biſecting the angle contained by the diameter paſſing through that point, and a ſtraight line drawn from the ſame point to the focus; that ſtraight line cuts the parabola in one other point, but not in more than one.

If from E, a point in the parabola, the ſtraight line EG be drawn, ſo as neither to be parallel to the axis, nor to biſect the angle Fig. 3. 4. n. 1. 2.

angle contained by the diameter EF and the ſtraight line EC; from the focus C let a perpendicular be drawn to EG, and let it meet the directrix in A; and, making A*f* equal to AF, through *f* draw *fe* parallel to FE, and let *fe* meet EG in *e*; the point *e* will be in the parabola.

Fig. 3. There are two caſes. The one is that in which EG paſſes through the focus: Becauſe EC is equal to EF, and each of the angles ECA, EFA a right angle; therefore AC is [5. and 6. 1. Elem.] equal to AF; and, conſequently, it is equal to A*f*; and each of the angles AC*e*, A*fe* is a right angle: *e*C is, therefore, equal to *ef*; and therefore the point *e* is in the [2. 1.] parabola.

Fig. 4. n. 1. 2. In the other caſe, EG paſſes not through the focus. From the centre E, at the diſtance EC, deſcribe a circle, meeting CA again in H; and deſcribe another circle through the points C, H, *f*; then, becauſe EC is equal to EF, and that EFA is a right angle, the circle deſcribed from the centre E touches [16. 3. Elem.] the directrix in F: therefore the rectangle CAH is equal to [36. 3. Elem.] the ſquare of AF, that is, to the ſquare of A*f*: therefore A*f* touches the circle [37. 3. Elem.] *f*CH; and the centre of this circle is [19. 3. Elem.] in *fe*; it is alſo in GE, which biſects CH at right angles: it is, therefore, in the point *e* where *fe*, GE interſect each other. *e*C, therefore, is equal to *ef*; and, therefore, the point *e* is in the [2. 1.] parabola.

It is evident, that EG cuts the parabola no where but in the points E, *e*: For, if poſſible, let EG cut it alſo in another point ϵ; and let $\epsilon\varphi$ be drawn perpendicular to the directrix AB; a circle, then, deſcribed from the centre ϵ, diſtance ϵC, paſſes through H, and touches the directrix in the point φ, at a diſtance from the point A, leſs or greater than that of the point F or *f* from A; and the ſquare of φA being equal to the rectangle CAH, is equal to the ſquare of FA: which is abſurd. Fig. 4. n. 1. 2. n. 1. 2.

COR. Of all the ſtraight lines that can be drawn from any point of a parabola, only one can touch the parabola; for the diameter through the point falls [4. 1.] within the parabola; and any other ſtraight line, except that which biſects the angle contained by the diameter through the point, and the ſtraight line drawn from the point to the focus, meets the parabola again in another point.

PROP. VIII. THEOR.

If from the focus of a parabola to a ſtraight line there is drawn a perpendicular meeting the directrix; if the ſegment of the perpendicular intercepted between the focus and the ſtraight line, is not greater than its other ſegment intercepted between the ſtraight line and the directrix; that ſtraight line, neceſſarily, meets the parabola.

Fig. 4 n. 1. 2. Let C be the focus of a parabola, and let CG, which meets the directrix in A, be a perpendicular to the ſtraight line LG; if the ſegment CG, between C and LG, be not greater than the ſegment GA, between LG and the directrix, the ſtraight line LG neceſſarily meets the parabola.

When the ſegments CG, GA are equal, it is plain, from what was demonſtrated in Prop. 6. that the ſtraight line LG touches the parabola in the point where the diameter through A meets the ſame LG.

But if CG be leſs than GA, take GH equal to GC; and from the point A, and on either ſide of A, place, in the directrix, AF, or A*f*, ſuch, that the ſquare of either may be equal to the rectangle CAH; and having deſcribed a circle through the points C, H, F, draw, through F, FE perpendicular to AF, and let FE meet LG in E: and the ſquare of AF being equal to the rectangle CAH, AF touches the circle in F; and therefore the centre of the circle is in FE: but as CH is biſected at right angles by the ſtraight line LG, the centre of the circle is likewiſe in LG: it is, therefore, in E, the point where FE, LG interſect each other: hence EC is equal to EF; and, of conſequence, the point E is in the parabola. In like manner, if *ef* drawn perpendicular to the directrix meets LG in *e*, the point *e* is in the parabola.

Cor. Hence any ſtraight line paſſing through a point within a parabola, meets the parabola.

Caſe 1. If the ſtraight line is a diameter, it is evident, from Prop. 4. 1. that it meets the parabola.

Caſe 2. When the ſtraight line is not a diameter. Let LG paſs through the point L within the parabola; it will meet the parabola. From the focus C let the ſtraight line CG be drawn at right angles to LG, and let it meet the directrix in A; and join LC, LA: then becauſe the point L is within the parabola; a ſtraight line drawn from L perpendicular to the directrix, is greater [3. 1.] than LC: LA, therefore, which is not leſs than this perpendicular, is greater than LC: AG, therefore, is greater than [47. 1. Elem.] GC; and therefore LG meets the parabola.

PROP. IX.

The angle contained by a diameter of a parabola, and a ſtraight line drawn from the vertex of that diameter to the focus, is biſected by the ſtraight line that touches the parabola in that vertex.

The angle ADC, contained by the diameter AD, and the ſtraight line DC, is biſected by DE, a ſtraight line touching the parabola in the vertex D: for if DE biſects not the angle ADC, it is poſſible for ſome other ſtraight line to do it; and this other ſtraight line alſo will touch the parabola: which is abſurd. Fig. 5.

COR. 1. On the other hand, if AD be a diameter, and ED touch the parabola in the vertex D of AD, and if the angle ADE be equal to the angle CDE, DC paſſes through the focus: Or, if DC paſſes through the focus, DE touching the parabola in D, and the angle ADE being equal to the angle CDE; DA is a diameter.

COR. 2. If from any point D which is in a parabola, but which is not the vertex of the axis, a ſtraight line DE be drawn touching the parabola, the angle EDA contained towards the directrix by the ſtraight line DE and the diameter DA, is leſs than a right angle; for the angle ADC, which is the double of EDA, is leſs than two right angles.

PROP. X. THEOR.

If from a point in a parabola a ſtraight line be drawn touching the parabola, and if from the ſame point a perpendicular be drawn to the axis; the ſegment of the axis intercepted between the perpendicular and the touching line, is biſected in the vertex of the axis.

Fig. 5. Let D be a point of a parabola, and let DE drawn from D touch the parabola, and DH be perpendicular to the axis; the ſegment EH of the axis is biſected in F, the vertex of the axis.

Through D let DA be drawn perpendicular to the directrix; let DC be drawn to the focus, and let the axis meet the directrix in B: And becauſe the angle GDE is equal to the angle ADE, that is, to the alternate angle CED, CE is equal to CD, or DA, that is, to HB; and CF is equal to FB; therefore the remainder FE is equal to the remainder FH.

PROP.

PROP. XI. THEOR.

Every ſtraight line parallel to a ſtraight line that touches the parabola, and terminated both ways by the parabola, is biſected by the diameter paſſing through the point of contact, that is, it is ordinately applied to this diameter.

The ſtraight line E*e*, which is terminated in the points E, *e*, being parallel to DK, a ſtraight line touching the parabola; and AD, the diameter which paſſes through the point of contact D, meeting E*e* in L; LE is equal to L*e*. Fig. 4. n. 1. 2.

Let AD meet the directrix AB in A; from the points E, *e* to the directrix, draw the perpendiculars EF, *ef*; and from the focus C draw CA meeting E*e* in G; and from the centre E, diſtance EC, deſcribe a circle meeting CA again in H; this circle will touch the directrix in F: join DC: then, becauſe DA is equal to DC, and the angle ADK equal to [9. 1.] CDK, DK is [4. 1. Elem.] perpendicular to AC: and, therefore, E*e* too is at right angles to the ſame AC: and becauſe E is the centre of the circle CFH, CG is equal to [3. 3. Elem.] GH: join *e*C and *e*H, and *e*C will be equal [4. 1. Elem.] to *e*H: A circle, therefore, deſcribed from the centre *e*, and at the diſtance *e*C, paſſes through H; and *e*C being equal to *ef*, it paſſes likewiſe through *f*: therefore, ſince the ſtraight line F*f* touches the circles, and the ſtraight line AHC cuts them, the ſquare of AF is equal to the [36. 3. Elem.] rectangle CAH, that is, to the ſquare of A*f*: therefore AF is equal to A*f*; and FE, AL, *fe*, are parallels: therefore LE is [2. 6. Elem.] equal to L*e*.

Cor. 1. On the other hand, if a ſtraight line E*e*, terminated both ways by a parabola, be biſected by the diameter AL, it is parallel to the tangent which paſſes through D, the vertex of AL: for if the ſtraight line touching the parabola in the point D, be not parallel to LE, let another ſtraight line be drawn touching the parabola, and parallel to LE; then the diameter which paſſes through the point where this other ſtraight line touches the parabola, biſects the ſtraight line E*e*: but, according to the hypotheſis, the ſame E*e* is biſected by the diameter AL: which is abſurd.

COR. 2. All ſtraight lines ordinately applied to any diameter, are parallel to one another.

COR. 3. If two or more parallels be terminated both ways by a parabola, the diameter which biſects the one, or one of them, biſects alſo the other, or the reſt of them: for the one that is biſected by a diameter, is parallel to the ſtraight line touching the parabola in the vertex of that diameter; and conſequently the other, or the others, is, or are, parallel to the ſame ſtraight line that touches the parabola in that vertex; and, therefore, is, or are, biſected by the ſame diameter.

COR. 4. Any ſtraight line, on the contrary, which biſects two parallels terminated both ways by a parabola, is a diameter: for if it is not, it is poſſible for ſome other ſtraight line biſecting one of the parallels to be a diameter; and being a diameter, this other ſtraight line muſt alſo biſect the other of them: but, according to the hypotheſis, the former of the ſtraight lines biſects both the parallels: which is abſurd. And if from the point of contact a ſtraight line be drawn biſecting another ſtraight line parallel to the tangent, and terminated both ways by the parabola, that ſtraight line is a diameter: for if it be not, let a diameter be drawn through the point of contact, this diameter muſt alſo biſect the parallel to the tangent: which is abſurd.

Cor. 5. And a ſtraight line drawn through the vertex of a diameter, ſo as to be parallel to ſtraight lines ordinately applied to that diameter, touches the parabola. This is manifeſt from cor. 1.

PROP. XII. Theor.

If from a point of a parabola a ſtraight line be drawn perpendicular to a diameter, and if from the ſame point a ſtraight line be ordinately applied to that diameter; the ſquare of the perpendicular is equal to the rectangle contained by the abſciſſa of the diameter and the *latus rectum* of the axis.

[*N. B.* An *abſciſſa* is the ſegment intercepted betwixt the vertex of a diameter and a ſtraight line ordinately applied to that diameter.]

Caſe 1. When the diameter is the axis of the parabola.

Let D be a point in a parabola, and DH a perpendicular to the axis BC; DH will be parallel to [5. 1.] the ſtraight line touching the parabola in the vertex of the axis; and therefore will be ordinately [11. 1.] applied to the axis: draw DC to the focus, and DA perpendicular to the directrix AB, and let F be the vertex of the axis: then, becauſe HB is equal to DA, that is, to DC, the ſquare of HB is equal to the ſquare of DC, that is, to the ſquare of DH, together with the ſquare of HC: but, ſince BF is equal to FC, the ſame ſquare of HB is equal to four times the rectangle HFC, together with the [8. 2. Elem.] ſquare of HC: therefore the ſquare of DH, together with the ſquare of HC, is equal to four times the rectangle Fig. 5.

HFB, together with the ſquare of HC: therefore the ſquare of DH is equal to four times the rectangle HFB, that is, to the rectangle contained by the abſciſſa HF, and the parameter of the axis.

Caſe 2. When the diameter to which the perpendicular is drawn is not the axis.

Fig. 4. n. 1. 2. Let EN be perpendicular to the diameter AD; let EL be an ordinate to AD, and D the vertex of the ſame AD; the ſquare of EN is equal to the rectangle contained by the abſciſſa LD and the parameter of the axis.

Draw DK parallel to LE; DK will therefore [5 cor. 11. 1.] touch the parabola in D: and let the ſame DK meet the axis in K; let EF be drawn at right angles to the directrix; and let a circle be deſcribed from the centre E, diſtance EF; and this circle will touch [cor. 16. 3. Elem.] the directrix in F, and paſs through the focus C: let AC be joined, and let it meet the circumference of the circle again in H, and the ſtraight lines DK, LE in the points P, G; and let LE meet the axis in O.

Becauſe the angles [9. 1. of this book, 4. 1. Elem.] CPK and CBA are right angles, and the angle BCP common, the triangles CBA, CPK are equiangular: AC, therefore, is [4. 6. Elem.] to CB, as CK to CP, that is, as [2. 6.; 16. 5. Elem.] OK to GP: the rectangle, therefore, contained by CA, GP is equal to [16. 6. Elem.] that contained by OK, CB: but becauſe CA is [9. 1. of this book, and 4. 1. Elem.] the double of CP, and CH the double of CG, AH is double of GP; and, conſequently, the rectangle CAH is equal to twice the rectangle CA, GP, that is, to twice the rectangle OK, CB: but the ſquare of EN, or of AF, is equal [3. 36. Elem.] to the rectangle CAH: it is, therefore, equal to twice the rectangle OK, CB, that is, to the rectangle contained by the abſciſſa LD, and the parameter of the axis.

COR. 1. Hence the ſquares of perpendiculars drawn from any points of a parabola to any diameters, are to one [1. 6. Elem.] another, as the abſciſſas intercepted between the vertices of thoſe diameters and the ordinates drawn from thoſe points.

COR. 2. The ſquares of ſtraight lines ordinately applied to the ſame diameter, are to one another, as the abſciſſas between thoſe ſtraight lines and the vertex of that diameter. Let EL, QR be ordinately applied to the diameter DN; and let EN, QS be perpendicular to the ſame: becauſe the triangle ELN is equiangular to the triangle QRS, the ſquare of EL is to that of QR, as the ſquare of EN to that of QS, that is, by the preceding corollary, as the abſciſſa DL to the abſciſſa DR.

COR. 3. And if from the vertices of two diameters there be drawn ſtraight lines ordinately applied to thoſe two diameters, that is, if the ſtraight line drawn from the vertex of each diameter be an ordinate to the other diameter, the abſciſſas between thoſe ordinates and the two vertices are equal to each other; for the perpendiculars drawn from the two vertices to the two diameters are equal.

PROP. XIII. THEOR.

If from a point of a parabola a ſtraight line be drawn ordinately applied to a diameter, the ſquare of half that ordinate is equal to the rectangle contained by the abſciſſa between that ſame ordinate and the vertex of that diameter, and the *latus rectum* of the ſame diameter.

Let AB be the directrix of a parabola, and AD a diameter, to which EL, drawn from the point E of the parabola, is ordinately applied; and through the vertex D of the diameter AD, draw DK parallel to EL; DK, of conſequence, will touch the parabola: draw DM perpendicular to the axis, and from Q, the vertex of the axis, draw QR ordinately applied to the diameter DL; and, conſequently, parallel to EL. Fig. 4. n. 1. 2.

Since QR is equal to DK, its ſquare is equal to [47. 1. Elem.] the ſquares of DM, MK; but the ſquare of DM is, by the firſt caſe of the foregoing propoſition, equal to four times the rectangle MQB: and ſince MQ is equal [10. 1.] to QK, the ſquare of MK is equal to four times the ſquare of MQ: therefore the ſquare of QR is equal to four times the rectangle MQB, together with four times the ſquare of MQ, that is, to four times [3. 2. Elem.] the rectangle QMB: but MQ, or QK, is equal to DR, and MB to DA; therefore the ſquare of QR is equal to four times the rectangle RDA: and ſince QR, EL are ordinately applied to the diameter AD, the ſquare of QR is to the ſquare of EL as [2 cor. of the preceding propoſition] RD to LD, that is, as four times the rectangle RDA to four times the rectangle LDA: but the ſquare of QR, as hath been proved, is equal to four times the rectangle RDA; therefore the ſquare of EL is equal to four times the rectangle LDA, that is, to the rectangle contained by the abſciſſa LD and the parameter of the diameter AD.

It was from the property above demonſtrated, that Apollonius named the curve line which is the ſubject of this book the PARABOLA.

COR. 1. If from a point E to AD, a diameter of the parabola, a ſtraight line EL is drawn parallel to ſtraight lines ordinately applied to the diameter AD, and meeting the ſame AD below its vertex D; if the ſquare of EL is equal to the rectangle contained by the abſciſſa LD and the parameter of the diameter AD; the point E is in the parabola.

For ſince the point L is within [4. 1.] the parabola, the ſtraight line EL neceſſarily meets [cor. 8. 1.] the parabola: if, therefore, EL does not meet the parabola in the point E, on the ſame ſide of the diameter with E, let it meet it, if poſſible, in ſome other point, nearer, or more remote, from the diameter, than E is: let this other point be ε; then the ſquare of εL is equal to the rectangle contained by LD, and the parameter of the diameter, that is, according to the hypotheſis, to the ſquare of EL; which is abſurd. Fig. 4. n. 2.

COR. 2. If from two points E, Q, one of which, Q, is in the parabola, there be drawn to the diameter AD ſtraight lines EL, QR, parallel to ſtraight lines ordinately applied to AD; if the ſquares of the parallels be to one another as the abſciſſas between the parallels and the vertex of the ſame AD; the other point E is alſo in the parabola.

If the diameter LD meets the directrix in A, four times AD is its *latus rectum*: then, ſince the ſquare of QR is to that of EL as RD is to LD, that is, as four times the rectangle RDA to four times the rectangle LDA; and ſince, by the propoſition, the ſquare of QR is equal to four times the rectangle RDA; the ſquare of EL is equal to four times the rectangle LDA; and therefore, by the preceding corollary, the point E is in the parabola.

PROP. XIV. Theor.

If a ſtraight line be drawn from a point of a parabola ſo as to be ordinately applied to a diameter, and if another ſtraight line be drawn from the ſame point ſo as to touch the parabola, and meet that diameter; the ſegment (of the diameter) intercepted betwixt the ordinate and the tangent is biſected in the vertex of the diameter.

Fig. 6. A being a point of a parabola; AC drawn from A, ſo as to be ordinately applied to the diameter BC, and AD drawn from the ſame point, ſo as to touch the

parabola,

parabola, and meet BC in D; the ſegment CD is biſected in B, the vertex of BC.

From the vertex B let BE be drawn parallel to AD; it will be ordinately [11. 1.] applied to the diameter AE, and the abſciſſa BC will be equal to the abſciſſa [3 cor. 12. 1.] AE, that is, to BD.

COR. 1. On the other hand, AC being ordinately applied to the diameter BC, if AD be drawn making BD equal to BC, AD touches the parabola.

For if AD does not touch the parabola, let AF touch it; FB then is equal to BC: which is impoſſible.

COR. 2. If a ſtraight line touches a parabola, its ſegment between the point of contact and any diameter is biſected by a ſtraight line touching the parabola in the vertex of that diameter.

Let AD, a ſtraight line touching the parabola, meet the diameter CB in the point D, and the tangent GB in G; let AC be drawn parallel to BG; AC will be ordinately applied to the [11. 1.] diameter CB: and ſince, by the propoſition, CB is equal to BD, AG is likewiſe equal to GD.

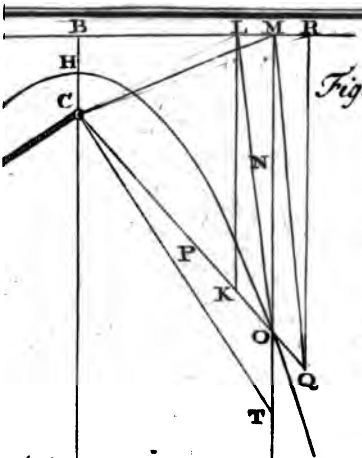
PROP. XV. PROB.

To find a diameter, the axis, the *latus rectum* of the axis, the focus, and the directrix of a parabola given in poſition.

Fig. 7. Let two parallel ſtraight lines AB, CD be drawn; let them be terminated in the parabola, and biſected in the points F, E; join FE, and let it meet the parabola in G: GF is [4. cor. 11. 1.] a diameter.

Next, in the diameter GF, and below its vertex G, take any point H; and through that point draw KHL perpendicular to the diameter GF, and meeting the parabola in the points K, L; and through M, the middle point of KL, draw MN parallel to the diameter GF, and meeting the parabola in N; and let NO be drawn parallel to MH: then, becauſe MN is parallel to GH, it is a diameter: but KL is ordinately applied to MN; therefore NO touches [5. cor. 11. 1.] the parabola in N. And becauſe MNO is a right angle, MN is [2. cor. 9. 1.] the axis: and a third proportional to NM, MK is the [13. 1.] *latus*

rectum



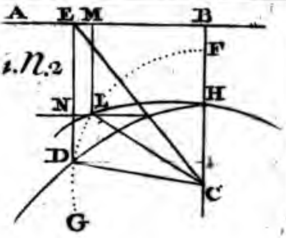
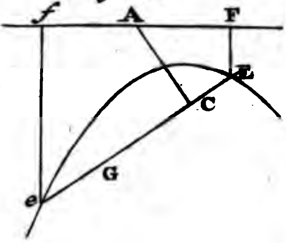


Fig. 1. N. 2

Fig. 3.



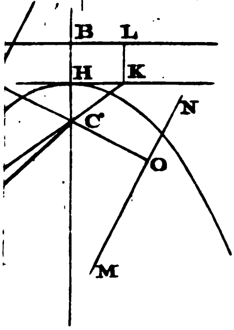
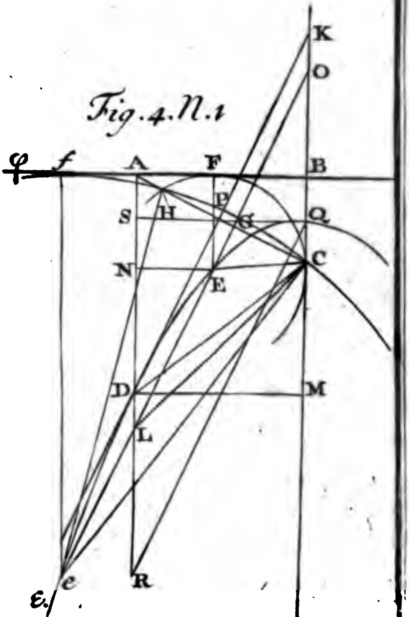


Fig. 4. N. 1



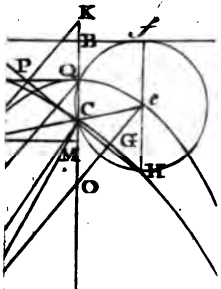


Fig. 4. N. 2

TAB. I. *pag. 14.*

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rectum of the axis: and the diſtance of the focus from the vertex of the axis is equal to a fourth part of the *latus rectum* of the axis; therefore the focus is given. After the ſame manner is the directrix found.

PROP. XVI. PROB.

The directrix and the focus of a parabola being given in poſition, to deſcribe the parabola.

Let AB be the directrix, and C the focus, and with a ruler and ſtring deſcribe the [def. 1.] parabola; or as many points of the parabola as may be thought neceſſary may be thus found; through the focus C draw CB at right angles to the directrix, and CB will be the axis: to the axis CB draw any perpendicular LG, meeting it below its vertex F, and in the ſame axis place CH equal to BG; CH will thus be greater than CG; and from the centre C, diſtance CH, deſcribe a circle, cutting the ſtraight line LG in D, *d*; theſe points are in the parabola. Fig. 8.

For the ſtraight line DA, drawn to the directrix, ſo as to be parallel to GB, is equal

qual to GB, that is, to CH, that is, to CD; and D, therefore, is [2. 1.] in the parabola. In the ſame manner it may be ſhewn, that *d* is in the parabola.

Cor. Hence, if the directrix AB of a parabola, and F, the vertex of the axis, be given in poſition, the parabola may be deſcribed, by drawing FB at right angles to the directrix, and making FC equal to FB; for C will be the focus. In like manner, if the vertex F and focus C be given; join CF, and produce it to B, ſo that FB may be equal to FC; a ſtraight line drawn through B at right angles to BC, will be the directrix: and if the axis GF, its vertex F, and its parameter FK, be given in poſition, the directrix may be found by making FB equal to a fourth part of the parameter FK, and drawing BA at right angles to the ſame FB. In like manner the directrix may be found, if the axis, the focus, and the parameter of the axis, be given in poſition. In all theſe caſes, therefore, the parabola may be deſcribed according to the propoſition.

PROP.

PROP. XVII. PROB.

The axis and its vertex, and a point without the axis and below its vertex, being given in poſition, to deſcribe the parabola which will paſs through that point.

The axis FH, its vertex F, and D a point without it, and below the vertex F, being given in poſition; it is propoſed to deſcribe the parabola which ſhall paſs through D. Fig. 8.

Having drawn from the point D, GD perpendicular to the axis, find [11. 6. Elem.] FK a third proportional to the two ſtraight lines FG, GD; then, taking FB equal to the fourth part of it, and making FC equal to FB, draw BA parallel to DG; and let a parabola be deſcribed, having C for its focus, and AB for the directrix; this parabola will paſs through the point D. For ſince FG, GD, FK are proportionals, the ſquare of GD is equal to the rectangle GFK: and FK is the parameter of the [def. 6. 1.] diameter FG; therefore the point D [1. cor. 13. 1.] is in the parabola.

PROP. XVIII. PROB.

Fig. 9. Two ſtraight lines AB, AC, which meet each other in the point A, being given in poſition, and a ſtraight line DE being given in magnitude; to deſcribe a parabola which may have AB for a diameter, and DE for the parameter of AB, and which the ſtraight line AC may touch in the point A.

Take of DE the fourth part DF, and in BA produced place AG equal to DF, and draw GH at right angles to AG; then, making the angle CAK equal to GAC, and the ſtraight line AK equal to AG, deſcribe a parabola, which may have K for the focus, and GH for the directrix; AB will be one of the diameters, and the paramater of AB will be equal to the [def. 4. 6.] quadruple of AG, that is, to DE: and ſince AG is equal to AK, the point A is in the parabola; and ſince the angle GAC is equal to CAK, the ſtraight line AC touches the parabola in the point A.

Cor. If from a point L, a ſtraight line LM be drawn in a given angle LMA, to a ſtraight line AB given in poſition; and if the ſquare of LM be equal to the rectangle contained by a given ſtraight line DE, and the ſegment MA, intercepted between the ſame LM and the given point A; the point L is in a parabola given in poſition. Having drawn through the point A a ſtraight line AC parallel to LM, deſcribe, according to the propoſition, a parabola which may have AB for a diameter, and DE for the parameter of AB, and which the ſtraight line AC may touch in the point A; this parabola is the *locus* of the points L: for ſince the ſquare of LM is, by hypotheſis, equal to the rectangle contained by MA and DE, and that LM is parallel to AC that touches the parabola, and conſequently to ſtraight lines ordinately applied to the diameter MA; the point L is in the [1. cor. 13. 1.] parabola.

PROP.

PROP. XIX. PROB.

Fig. 9. A diameter AB, and its vertex A, being given in poſition, and a ſtraight line LM, which meets AB in M below the vertex, being given in poſition and magnitude; to deſcribe a parabola which may paſs through the point L, and in which the ſtraight line LM may be ordinately applied to the diameter AB.

Through the vertex A draw AC parallel to LM, and let DE be a third proportional to AM, ML; and, according to the preceding propoſition, deſcribe a parabola which may have AB for a diameter, and DE for the parameter of AB, and which AC may touch in the point A: then, becauſe ML is parallel to AC, it is ordinately applied to the diameter AB; and becauſe AM, ML, DE are proportionals, the ſquare of ML is equal to the rectangle contained by the abſciſſa AM, and DE the parameter of the diameter AB; and therefore the point L is [1. cor. 13. 1.] in the parabola.

PROP. XX. PROB.

A diameter of a parabola, and the vertex of that diameter, being given in poſition, and the *latus rectum* of the ſame diameter being given in magnitude, and a point in the parabola being given; to deſcribe the parabola.

Let AB be the diameter given in poſition, and A its vertex; in AB, and above the vertex A, place the ſtraight line AC equal to the given *latus rectum*; and let D be the given point in the parabola. Suppoſe what is required done; and let AD be the parabola to be deſcribed: and having drawn the ſtraight line AE touching it in A, and meeting the diameter drawn through D in the point E, complete the parallelogram AEDF: therefore DF is ordinately applied to the diameter AB; and therefore the ſquare of DF, or AE, is equal to the rectangle FAC: as, therefore, Fig. 10.

FA or DE to AE, ſo is AE to AC; and they contain the equal [29. 1. Elem.] angles DEA, EAC; therefore the triangle DEA is equiangular to the triangle EAC: and thus the angle AEC is equal to the angle EDA, or FAD. If, then, upon AC a ſegment of a circle be deſcribed, containing an angle equal to FAD, the point E will [converſe of 21. 3. Elem.] be in the circumference of this ſegment. But the angle FAD is given, becauſe FA, DA are [26. dat.] given in poſition; therefore the angle AEC is given: and AC is given in poſition and magnitude; therefore the ſegment AEC [8. def. dat.] is given in poſition. The point E, then, is in the circumference of a circle given in poſition: but it is alſo in the ſtraight line DE which is given in poſition; the point E, therefore, is given: and the point A is given: therefore the ſtraight line AE is given in poſition. It is poſſible, therefore, to deſcribe [18. 1.] a parabola which may have AB for a diameter, and AC for the *latus rectum* of AB, and which AE may touch in the point A.

In order to the compoſition, it is required, that a ſegment of a circle containing an angle equal to FAD, be deſcribed upon AC, and that a ſtraight line drawn through the point D, parallel to AB, meet the circumference of that ſegment. But theſe conditions it is ſometimes impoſſible to fulfil. Hence the problem cannot always be ſolved.

When the ſtraight line drawn through D parallel to AB, is a tangent to the ſegment, the problem admits of only one ſolution *. The parabola, and *latus rectum* of the diameter AB, which ſolve this caſe, are determined, if a parabola be found having AB for a diameter, and A for the vertex of AB, and which paſſes through the point D; and if the *latus rectum* of BA be ſuch, that a ſegment of a circle deſcribed upon it, when placed above A, and in the direction of AB, may contain an angle equal to BAD; and that a ſtraight line drawn through D, ſo as to be parallel to AB, may touch the circumference of that ſegment: ſuppoſe what is required done: let AG be the parameter of the diameter AB; and upon AG let the ſegment of a circle containing an angle equal to the angle BAD, or ADE, be deſcribed; and let the ſtraight line DE parallel to AB touch the circumference of that ſegment in H, and join AH, GH: ſince, then, the angle AHD is equal [32. 3. Elem.] to AGH in the oppoſite ſegment, and that, according to the hypotheſis, the angle ADH is equal to AHG; the triangles ADH, AHG are equiangular: the angle DAH is, therefore, equal to HAG: but the angle DAG is given; and, conſequently, its half DAH is given: and the ſtraight line AD is given in poſition; therefore AH alſo is [29. dat.] given in poſition: the point H too is given, where AH meets DE, given in poſition; and the angle AHG is given: hence HG is given [29. dat.] in poſition; and therefore the point G is given: hence the ſtraight line AG is given in magnitude: ſince, then, AG in the parabola, which paſſes through the point D, is the *latus rectum* of the diameter AB, of which A is the vertex; ſince a ſtraight line touching the parabola in the vertex of the diameter AB, meets, as hath been proved, the diameter drawn through D, in the point where this diameter meets the circumference of the circle, the ſegment of which, deſcribed upon the *latus rectum* of the diameter, paſſing through A, contains an angle equal to ADH; and ſince, in the preſent caſe, the diameter DE meets the circumference of this ſegment in H; HA touches the parabola in A: and AB, AH being given in poſition, and AG given in magnitude, the parabola, according to the 18th propoſition, can be deſcribed.

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* In all caſes in which the ſtraight line drawn through D parallel to AB cuts the circumference of the ſegment in two points, the problem admits of two ſolutions.

The compoſition of this caſe is as follows: Join AD, and through D draw DE parallel to AB; to DE draw AH, biſecting the angle DAG; and through H to AB draw HG, making the angle AHG equal to ADH or DAB; and let a parabola be deſcribed which may have AB for a diameter, and AG for the *latus rectum* of AB; and which AH may [18. 1.] touch in A: this parabola will paſs through D, and DH will touch the circle deſcribed about AHG. Draw DK parallel to AH; and ſince the triangles DAH, AHG are iſoſceles and equiangular, DH, HA, AG, and conſequently KA, KD, AG, are proportionals; the ſquare of DK is, therefore, equal to the rectangle KAG; and DK is parallel to the tangent AH: hence the point D is in the [1 cor. 13. 1.] parabola: and becauſe the angle AHD is equal to AGH in the oppoſite ſegment, DH touches [conv. 32. 3. Elem.] the circle in H.

It remains to be inquired, whether the parameter AG be greater or leſs than the parameter of the diameter AB in any other parabola, having AB for a diameter, and A for the vertex of AB, and which paſſes through D. Let there be any other parabola admitting of theſe conditions; and let AE touch it in A, and meet the diameter paſſing through D in the point E, and the circle GHA in L: having joined LG, draw EC parallel to it; draw alſo DF parallel to EA; DF, therefore, is ordinately applied to the diameter AB: and becauſe the angle ADE is equal to the angle AHG or ALG, that is, to AEC, and that the angle DEA is equal to EAC, the triangle EDA is equiangular to the triangle AEC: therefore the ſtraight lines DE, EA, AC, that is, AF, FD, AC are proportionals; the ſquare of DF is, therefore, equal to the rectangle FAC: and, for this reaſon, AC is the parameter of the diameter AB in this parabola. And becauſe DE touches the circle in H, AL is leſs than AE; and

therefore

therefore AG is leſs than AC: therefore AG is the leaſt of all the poſſible parameters of the diameter AB, in parabolas which have AB for a diameter, and A for the vertex of AB, and which paſs through D. After the ſame manner it may be ſhewn, that in any parabola whatever, which anſwers the conditions of the propoſition, the *latus rectum* of the diameter AB is greater or leſs, according as the tangent drawn through A, and ſituated on either ſide of the tangent AH, is more remote from, or nearer to, the ſame AH.

To proceed to the compoſition of what was analyſed in the firſt caſe: If the propoſed parameter be equal to AG, found in the manner above mentioned, the parabola, as hath been ſhewn, may be deſcribed, and will be the only one that can fulfil what is required in the problem. If, next, the propoſed parameter be leſs than AG, it is impoſſible to conſtruct the problem: or if the propoſed parameter, for example, AC, be greater than AG; upon AC deſcribe the ſegment of a circle containing an angle equal to ADH, or DAB; and ſince DH touches the circle AHG, it muſt cut the ſegment deſcribed upon AC in two points: let E be one of them; and join AE; and let a parabola [18. 1.] be deſcribed, having AB for a diameter, and AC for the parameter of AB, and to which the ſtraight line AE may be a tangent; and draw DF parallel to AE; then it may be ſhewn, as above, that DE, EA, AC, that is, that AF, FD, AC, are proportionals; that the ſquare of DF is, therefore, equal to the rectangle FAC, contained by the abſciſſa FA and the parameter AC; and that, conſequently, the parabola paſſes through the point D. The ſame thing may be demonſtrated with regard to the other parabola, which has for a tangent the ſtraight line joining A, and the other point of interſection *e*. And as, in the inveſtigation of the problem, it has been proved, that the angles GAH, HAD are equal; the angle AKD is, therefore, equal to ADK; and, of conſequence, AK is equal to AD: but AG is a third proportional to AK, KD, or to AD, DK; that is, the leaſt parameter is a third proportional to the ſtraight line which joins the vertex of the diameter given in poſition and the given point; and the ſtraight line which is drawn from the ſame point to the diameter, ſo as to cut off from the diameter a ſegment equal to the firſt proportional.

The firſt nine definitions in the firſt book of Apollonius of Perga's Conic Sections.

1. VIII. If a ſtraight line joining any point and the circumference of a circle not in the ſame plane with the point, be produced from the point in the oppoſite direction, and then, while the point remains fixed, be carried round in the direction of that circumference till it return to the place from which the motion has commenced; by the revolution of that ſtraight line, a ſurface, named *the conical ſurface*, and which conſiſts of two ſurfaces connected together at the fixed point, will be deſcribed. The two connected ſurfaces may each of them be infinitely increaſed, if the ſtraight line with which they are deſcribed be produced both ways to an infinite diſtance.

2. IX. The fixed point is named *the vertex* of the conical ſurface.

3. X. The ſtraight line drawn through the point and the centre of the circle is named *the axis.*

4. XI. The figure contained by the circle, and the ſurface which is intercepted between the vertex and the circumference of the circle, is named *the cone.*

5. XII. The ſame fixed point, which is the vertex of the ſurface of the cone, is named *the vertex of the cone.*

6. XIII. The ſtraight line drawn from the vertex to the centre of the circle, is named *the axis of the cone.*

7. XIV. And the circle itſelf is named *the baſe of the cone.*

8. XV. Cones which have their axis at right angles to the baſe, are named *right-angled cones.*

9. XVI. And cones which have not their axis at right angles to the baſe, are named *ſcalene cones.*

PROP. XXI. [*Prop.* 1. *B.* 1. *Apoll.*]

Straight lines drawn from the vertex of the ſurface of a cone to points in that ſurface, are in that ſame ſurface.

Let

Fig. 6.

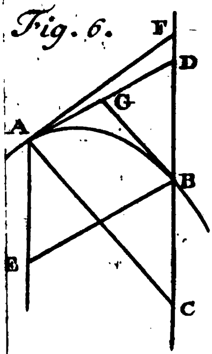


Fig. 7.

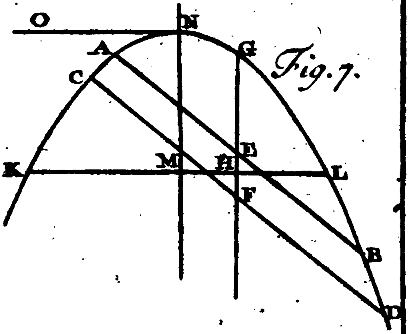
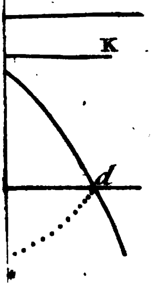
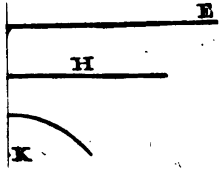


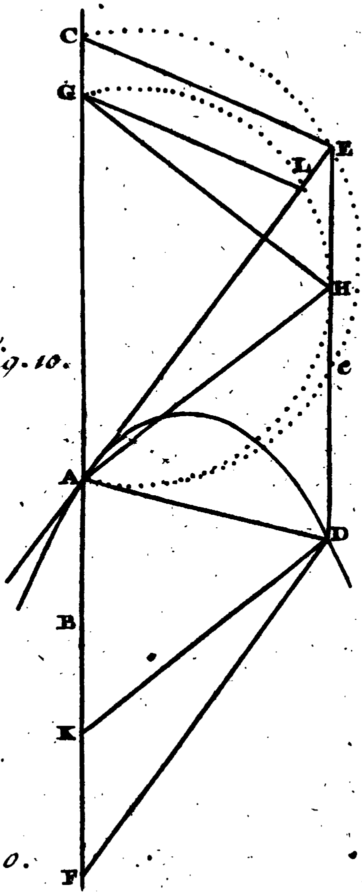
Fig. 8.





9.

Fig. 10.



TAB. II. pag 20.

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Let there be the ſurface of a cone: let A be its vertex; and having taken any point B in that ſurface, join ACB: the ſtraight line ACB is in that ſame ſurface. Fig. 11.

For, if poſſible, let ACB not be in the ſurface of the cone; and let DE be the ſtraight line with which the cone is deſcribed, and the circle EF the baſe; and if DE be revolved in the circumference of EF, it will paſs through the point B; and thus two ſtraight lines ACB, AGB will have the ſame extremities: which is abſurd. Therefore the ſtraight line drawn from the point A to B, is not without the conical ſurface; therefore it is in that ſurface.

COR. A ſtraight line drawn from the vertex of a cone to any point within the ſurface, falls within the ſurface; but if drawn from the vertex to any point without the ſurface, it falls without the ſurface.

PROP. XXII. [*Prop.* 3. *B.* 1. *Apoll.*]

If a cone be cut by a plane paſſing through its vertex, the ſection is a triangle.

Fig. 12. Let there be a cone which has the point A for its vertex, and the circle BDC for its baſe; let it be cut through the point A by any plane; and let the ſections made in the ſurface be the lines AB, AC, and the ſection in the baſe the ſtraight [3. 11. Elem.] line BC; ABC is a triangle.

For ſince the ſtraight line drawn from the point A to B, is both in the cutting plane and in [21. 1.] the conical ſurface, it is the common ſection of the two; therefore the ſection AB is a ſtraight line: for a like reaſon, the ſection AC is a ſtraight line; and BC too is a ſtraight line: therefore ABC is a triangle.

PROP. XXIII. [*Prop.* 4. *B*, 1. *Apoll.*]

If either of the ſurfaces at the vertex be cut by a plane parallel to the circle in the circumference of which the ſtraight line that deſcribes the conical ſurface is revolved; the plane incloſed within the ſurface is a circle having its centre in the axis; and the figure contained by this circle, and that part of the conical ſurface which is intercepted between the cutting plane and the vertex, is a cone.

Let there be a conical ſurface the vertex of which is A, BC being the circle in the circumference of which the ſtraight line which deſcribes the *ſurface* is revolved; let it be cut by any plane parallel to the circle BC, and let this plane make in it a ſection DLE: the line DLE is the circumference of a circle the centre of which is in the axis. Take the centre of the circle BC, and let it be F; join AF; AF, conſequently, is [def. 10.] the axis, and meets the cutting plane; let it meet it in G; next, let any plane paſs through the ſame AF; and the ſection made by this plane Fig. 13.

plane will be [22. 1.] a triangle ABC. And becauſe the points D, G, E are both in the cutting plane DLE, and in the plane ABC, DGE is [3. 11. Elem.] a ſtraight line. Again, in the line DLE take any point H; join AH, and produce it; AH then will meet the circumference BC; let it meet it in K, and join GH, FK: And becauſe the two parallel planes DLE, BC are cut by the plane ABC, their [16. 11. Elem.] common ſections with it are parallels: DE, conſequently, is parallel to BC: and, for the ſame reaſon, GH is parallel to FK: therefore, as AF to [4. 6. Elem.] AG, ſo is FB to GD, FC to GE, and FK to GH; and the three ſtraight lines BF, KF, CF, are equal; therefore the three ſtraight lines DG, GH, GE [14. 5. Elem.] are alſo equal. After the ſame manner it may be demonſtrated, that any other ſtraight lines whatever, drawn from the point G to the line DLE, are equal. The line DLE is, therefore, the circumference of a circle having its centre G in the axis.

Cor. The figure contained by the circle DLE, and that part of the conical ſurface which is intercepted between this circle and the point A, is a cone; and the common ſection of the cutting plane and the triangle paſſing through the axis, is a diameter of the circle DLE.

PROP. XXIV. [*Prop.* 5. *B.* 1. *Apoll.*]

If a ſcalene cone cut through the axis by a plane at right angles to the baſe, be cut alſo by another plane at right angles to the triangle paſſing through the axis; if this other plane cuts off, towards the vertex, a triangle ſimilar to the triangle through the axis, but ſubcontrarily ſituated; the ſection made in the cone by this other plane is a circle.

Let there be a ſcalene cone, the vertex of which is the point A, and the baſe the circle BC; let it be cut through the axis by a plane perpendicular to the baſe, and let the ſection be the triangle ABC; let it be Fig. 14.

be alſo cut by another plane at right angles to the triangle ABC; and let this other plane cut off, towards the vertex, the triangle AGK ſimilar to the triangle ABC, but * ſubcontrarily ſituated; and let the ſection made in the ſurface be the line GKH: this line is the circumference of a circle.

In the lines GHK, BC take certain points H, L, from which let perpendiculars be drawn to the plane of the triangle ABC; theſe perpendiculars will [38. 11. Elem.] fall on the common ſections of the planes: accordingly, let them be HF, LM: HF, therefore, is parallel to [6. 11. Elem.] LM: next, through F draw DFE parallel to BC: the plane, therefore, which paſſes through FH, DE is parallel to the [15. 11. Elem.] baſe of the cone; and, for this reaſon, the ſection DHE is [23. 1.] a circle, of which DE is a diameter: the rectangle, therefore, contained by DF, FE is equal to the ſquare of FH. And ſince ED is parallel to BC, the angle ADE is equal to the angle ABC; and the angle AKG is placed equal to the angle ABC; therefore the

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* The meaning is this: The baſe GK is to be ſo placed, that it may make the angle AKG, and not the angle AGK, equal to the angle ABC.

angle AKG is alſo equal to ADE: and the angles at F are equal, for they are oppoſite vertical angles; therefore the triangle DFG is ſimilar to the triangle KFE: therefore as EF to FK, ſo is GF to FD; therefore the rectangle EFD is equal to the rectangle KFG. But the rectangle EFD, (that is, the rectangle contained by DF, FE,) has been proved to be equal to the ſquare of FH; therefore the rectangle contained by KF, FG is equal to the ſame ſquare of FH. It may, in like manner, be demonſtrated, that the ſquare of any ſtraight line whatever, drawn from the line GHK, ſo as to be perpendicular to GK, is equal to the rectangle contained by the ſegments into which that ſtraight line divides the ſame GK: the ſection GHK is, therefore, a circle having GK for a diameter *. — A ſection of this kind may be named *a ſubcontrary ſection.*

PROP. XXV.

If a cone cut through the axis by a plane, be cut likewiſe by another plane, cutting its baſe in the direction of a ſtraight line perpendicular to the baſe of the triangle paſſing through the axis; and if the common ſection of the triangle through the axis, and of the plane cutting the baſe of the cone in the direction of the perpendicular, be parallel to one of the ſides of the triangle through the axis; the line which is the common ſection of the baſe-cutting plane, and the conical ſurface, is a parabola having for a diameter the ſtraight line which is the common ſection of the triangle through the axis, and the ſame baſe-cutting plane.

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* See the *lemma* placed at the end of this book.

Fig. 15. Let there be a cone, the vertex of which is the point A, and the baſe the circle BC; let it be cut through the axis by a plane, and let the ſection be the triangle ABC;

let it be alſo cut by another plane, cutting its baſe in the direction of the ſtraight line DE perpendicular to the ſtraight line BC; let the line DFE be the ſection made in its ſurface; and let FG, the common ſection of the triangle through the axis, and that other plane, be parallel to AC, one of the ſides of that triangle; the line DFE is a parabola, and FG one of its diameters.

In the ſection DFE take any point H, and through H draw HK parallel to DE; produce HK to FG; and through K draw LM parallel to BC: therefore the plane paſſing through HK, LM is [15. 11. Elem.] parallel to the plane through DE, BC, that is, to the baſe of the cone: and, conſequently, the plane through HK, LM is a [23. 1.] circle of which LM is a diameter. But HK is perpendicular to LM, [10. 11. Elem.], becauſe DE is perpendicular to BC: therefore the rectangle LKM is equal to the ſquare of HK [35. 3. Elem.]; and, in like manner, the rectangle BGC is equal to the ſquare of DG: therefore the ſquare of DG is to the ſquare of HK, as the rectangle BGC to the rectangle LKM; and GC is equal to KM; therefore the rectangle BGC is to the rectangle LKM, as BG to

LK, that is, as GF to KF: therefore the ſquare of DG is to the ſquare of HK as the ſtraight line GF to the ſtraight line KF. Let, therefore, a parabola [19.1.] be deſcribed, which may have GF for a diameter, and F for the vertex of GF, and in which DG may be ordinately applied to the ſame GF: and becauſe the point D, by conſtruction, is in the parabola deſcribed, the point H is likewiſe in this ſame parabola [2 cor. 13. 1.]. And the ſame thing may be demonſtrated with regard to all the points of the ſection DFE.

The ſecond Lemma of Pappus, as it is extant in the firſt book of Apollonius's Conic Sections.

Fig. 16. Let ABC be a line, and let AC be a ſtraight line given in poſition; and let all the ſtraight lines drawn from the line ABC, ſo as to be at right angles to AC, be ſuch, that each of them may have its ſquare equal to the rectangle contained by the ſegments,

Fig. 12.

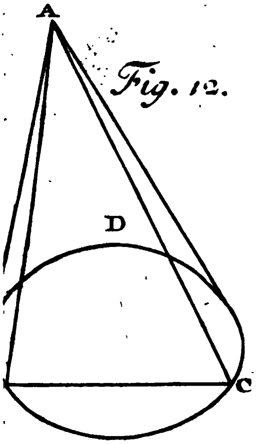


Fig. 13.

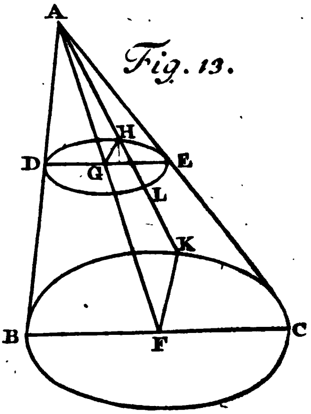


Fig. 16.

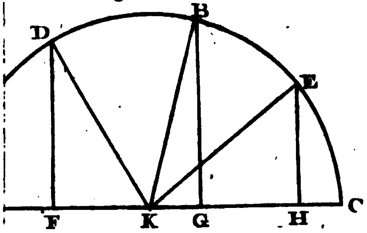
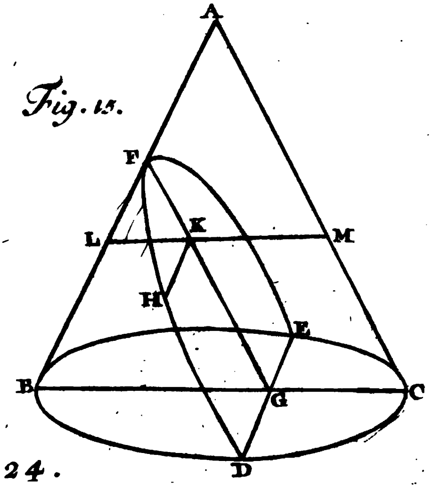


Fig. 15.



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into which it cuts AC; ABC is the circumference of a circle, and AC a diameter of that circle.

From the points D, B, E draw perpendiculars DF, BG, EH: then the ſquare of DF is equal to the rectangle AFC, the ſquare of BG to the rectangle AGC, and the ſquare of EH to the rectangle AHC. Biſect AC in K, joining KD, KB, KE: then, ſince the rectangle AFC, together with the ſquare of FK, is equal [5. 2. Elem.] to the ſquare of AK, and that the ſquare of DF is, by hypotheſis, equal to the rectangle AFC; the ſquare of DF, together with the ſquare of FK, is equal to the ſquare of AK: therefore the ſquare of DK [47. 1. Elem.] is equal to the ſame ſquare of AK: AK, therefore, is equal to KD. In like manner, each of the ſtraight lines BK, EK may be proved to be equal to AK or KC; therefore ABC is the circumference of a circle which has the point K for its centre, and is deſcribed about AC as a diameter.

ELEMENTS

OF THE

CONIC SECTIONS.

BOOK II.

Of the ELLIPSE.

DEFINITIONS.

I. IN two points D, E, taken in a plane, are fixed the ends of a ſtring: the length of the ſtring is greater than the diſtance between theſe points. A pin H applied to the ſtring, and held ſo as to keep it uniformly tenſe, is moved round, till it return to the place from which the motion has commenced; The point of the pin, as it moves round, deſcribes upon the plane a line named the ELLIPSE.

II. The points D, E are named the *foci*.

III. The point C which biſects the ſtraight line between the foci, is named the *centre* of the ellipſe.

IV. A ſtraight line paſſing through the centre, and terminated both ways by the ellipſe, is named a *diameter*; and the points where a diameter meets the ellipſe, are named the *vertices* of that diameter.

V. The diameter which paſſes through the foci, is named the *greater axis*.

VI. The diameter perpendicular to the greater axis, is named the *leſs axis*.

VII. Two diameters, each of which biſects all ſtraight lines in the ellipſe that are parallel to the other, are named *conjugate diameters*.

VIII. A ſtraight line not paſſing through the centre, but terminated both ways by the ellipſe, and biſected by a diameter, is ſaid to be *ordinately applied* to that diameter, or it is named, ſimply, an *ordinate* to that diameter. Alſo a diameter parallel to a ſtraight line ordinately applied to another diameter, is ſaid to be *ordinately applied* to that other diameter.

IX. A third proportional to two conjugate diameters is named the *latus rectum*, or the *parameter*, of that diameter, which is the firſt of the three proportionals.

X. A ſtraight line which meets the ellipſe only in one point, is ſaid to touch it in that point.

PROP. I. THEOR.

If two ſtraight lines be drawn from any point in an ellipſe to the foci, they are together equal to the greater axis.

Fig. 1. HD, HE drawn from H, a point in an ellipſe, to the foci D, E, are together equal to AB the greater axis.

Becauſe H is a point in the ellipſe, HD, HE are together equal to the length of the ſtring with which it is deſcribed; and becauſe the point A is likewiſe in the ellipſe, DA, EA are together equal to the length of the ſtring. For a like reaſon, EB, DB are together equal to the ſame length: DA, EA are, therefore, equal to EB, DB. Take away the common part DE, and the remainder, to wit, twice AD, will be equal to the remainder twice EB: AD, therefore, is equal to EB. Add the common part AE; and AD, together with AE, will be equal to the greater axis: but AD, together with AE, is equal to the length of the ſtring, that is, to HD together with HE; therefore HD and HE are together equal to the greater axis AB.

COR. 1. The greater axis is biſected in the centre C. For ſince [def. 3.] DC is equal to EC, and that DA is equal to EB; AC is equal to CB.

COR. 2. Two ſtraight lines drawn from a point without an ellipſe to the foci, are together greater than the greater axis; but if drawn from a point within an ellipſe to the foci, they are [20. 1. Elem.] together leſs than that axis.

COR. 3. A point is either in, without, or within, an ellipſe, according as two ſtraight lines drawn from it are either equal to, greater, or leſs, than the greater axis.

COR. 4. The diſtance of either vertex G of the leſs axis from either of the foci, is equal to half the greater axis. Join GD, GE: then, becauſe CD is equal to CE, CG common, and the angles at C right angles; the triangle CDG is equal to the triangle CEG; therefore DG is equal to EG: but DG and EG are together equal to the greater axis; therefore each of them is equal to the half of it.

Cor. 5. The leſs axis FG is biſected in the centre. Draw ſtraight lines DF, DG from the focus D to the vertices of the leſs axis; DF, DG, by the preceding corollary, will be equal; the angle DFC is, therefore, equal to DGC, and DCF, DCG are right angles; therefore CF is [26. 1. Elem.] equal to CG.

PROP. II. Theor.

The ſquare of half the leſs axis is equal to the rectangle contained by the ſegments of the greater axis, intercepted between the vertices of the greater axis and either of the foci.

Fig. 1. From either focus E to either vertex G of the leſs axis, draw the ſtraight line GE; let C be the centre of the ellipſe, and A, B the vertices of the greater axis: then the ſquares of GC, CE are together equal to the ſquare of GE, that is, to the ſquare

[4 cor. 1. 2.] of CB, that is, to the rectangle AEB together with the ſquare of EC [5. 2. Elem.]: take away the common ſquare of CE, and the remaining ſquare of GC will be equal to the remaining rectangle AEB.

PROP. III. THEOR.

Every diameter of an ellipſe is biſected in the centre.

Let HK be a diameter; it is biſected in C: for if CK be not equal to CH, let C*k* be equal to CH; and from the points H, K, *k* draw ſtraight lines to the foci D, E: then, becauſe CD is equal to CE, and that G*k* is placed equal to CH; the triangle DCH is equal to the triangle EC*k*, and the baſe DH to the baſe E*k*. EH, in the ſame manner, is equal to D*k*: therefore EK, DK are together equal to DH and HE together, that is, to EK and DK together: which is [21. 1. Elem.] abſurd; therefore CH is equal to CK. Fig. 2.

PROP. IV. THEOR.

If a ſtraight line be drawn from a point in an ellipſe, ſo as to be at right angles to the greater axis; if another ſtraight line be drawn from the ſame point to the nearer (to this point) of the foci; half the greater axis is to the diſtance of this focus from the centre, as the diſtance between the centre and the perpendicular is to the exceſs of half the greater axis above the ſtraight line drawn to this ſame focus.

Fig. 1. From H, a point in an ellipſe, let HK be drawn perpendicular to the greater axis AB, HE being drawn from the ſame point to the focus E: CB, the half of the greater axis, is to GE, the diſtance of the focus E from the centre, as CK, the diſtance between the centre C, and the perpendicular HK, is to the exceſs of CB above HE.

Having made BL equal to EH, and drawn HD to the other focus, deſcribe from the centre H, diſtance HE, a circle meeting the axis AB again in O, and the ſtraight line DH in the points M, N: then, becauſe DE is the double of CE, and OE the double of [3. 3. Elem.] KE, the whole, or the remainder DO, is double the whole, or the remainder CK: and becauſe DN is equal to DH together with HE, that is, to [1. 2.] AB, DN is the double of CB: but MN is the double of HE or LB; therefore the remainder DM is double the remainder CL: and becauſe of the circle, the rectangle NDM is [cor. 36. 3. Elem.] equal to the rectangle EDO; therefore, as [16. 6. Elem.] ND or AB to DE, ſo is DO to DM: but the halves of magnitudes have the ſame ratio to one another which the wholes have: as therefore CB to CE, ſo is CK to CL, the exceſs of CB above LB, or HE.

Fig. 1. n. 1.

COR. If in a ſtraight line AB given in poſition and magnitude, and biſected in C, a point E be taken, between the points B, C; and from the point H, KH be drawn perpendicular to AB; and HE be joined, and BL placed equal to it towards the point C; and CL, CK, CE, BC be proportionals, and L, K be on the ſame ſide of C; the point H is in an ellipſe given in poſition, that is, in an ellipſe that has AB for the greater axis, and the point E for one of the foci.

Make AD equal to BE, and, as directed in the firſt definition, deſcribe an ellipſe that ſhall have AB for the greater axis, and the points D, E for the foci; the point H will be in that ellipſe: for, if not, let KH meet the ellipſe in the point Q; join EQ, and make BR equal to it: therefore, by the propoſition, as CR to CK, ſo is CE to CB: but, by hypotheſis, CL is to CK as CE to CB; therefore CR is to CK as CL to CK: CL, therefore, is equal to CR; which is abſurd: the ellipſe, therefore, meets not the ſtraight line KH in Q; nor, as may in like manner be proved, does it meet HK any where on the ſame ſide of AB but in the point H. It therefore paſſes through the point H.

PROP.

PROP. V. THEOR.

The ſame conſtruction remaining, if there be placed from the nearer (to the point taken in the ellipſe) of the vertices of the greater axis, and towards the centre, a ſtraight line, equal to the diſtance between the point in the ellipſe and the nearer of the foci; the ſquare of the perpendicular is equal to the exceſs of the rectangle contained by the ſegments into which the ſame perpendicular divides the greater axis, above the rectangle contained by the ſegments intercepted between the point where the placed line terminates towards the centre, and the foci.

From B, the nearer (to the point H) of the vertices of the greater axis, and towards the centre C, let BL be placed equal Fig. 1. n. 1.2.

qual to HE; the ſquare of the perpendicular HK is equal to the exceſs of the rectangle AKB, contained by the ſegments into which HK divides the greater axis, above the rectangle DLE, contained by the ſegments intercepted between L, where BL terminates, and the foci D, E.

For ſince the ſtraight line CB is cut into any two parts in the point L, the ſquares of BC, CL are together equal [7. 2. Elem.] to twice the rectangle BCL, together with the ſquare of BL, that is, to twice [preced. prop. and 16. 6. Elem.] the rectangle ECK, together with the ſquare of BL, or HE, that is, to twice the rectangle ECK, together with the ſquares of KE, KH, that is, to the [7. 2. Elem.] ſquares of EC, CK, and KH together: the ſquares, therefore, of BC, CL are together equal to the ſquares of EC, CK, and KH together: but the ſquares of BC, CL are together equal [5. 2. and 2. ax. Elem.] to the rectangle AKB, together with the ſquares of CK, CL; and the ſquares of EC, CK, and KH are together equal [5. 2. and 2. ax. Elem.] to the rectangle DLE, together with the ſquares of CL, CK, KH; therefore the rectangle AKB, together with the ſquares of CK, CL, are equal to the rectangle DLE, together with the ſquares of CL, CK, KH: from theſe equals take away the common ſquares of CK, CL, and there will remain the rectangle AKB, equal to the rectangle DLE, together with the ſquare of HK; therefore the ſquare of KH is the exceſs of the rectangle AKB above the rectangle DLE.

PROP. VI. THEOR.

If a ſtraight line be drawn from a point of an ellipſe, ſo as to be parallel to either axis, and to meet the other axis; the ſquare of the axis it meets is to the ſquare of the other to which it is parallel, as the rectangle contained by the ſegments of the axis it meets is to its own ſquare.

In the ellipſe let H be the point from which the ſtraight line is drawn parallel to either axis, &c. Fig. 1. n. 1. 2.

Firſt, let HK, drawn from the point H,

be parallel to the leſs axis FG, and meet AB the greater in K; the ſquare of AB is to the ſquare of FG as the rectangle AKB to the ſquare of HK. Having drawn to the foci D, E two ſtraight lines HD, HE, place, from the vertex of the greater axis, and towards the centre C, the ſtraight line BL equal to HE the leſſer of them; and becauſe CB, CE, CK, CL are [4. 2.] proportionals, their ſquares are alſo proportionals: but the ſquare of CB is [5. 2. Elem.] made up of the ſquare of CK and the rectangle AKB; and the ſquare of CE of the [5. 2. Elem.] ſquare of CL and the rectangle DLE; therefore the ſquare of CB is [19. 5. Elem.] to the ſquare of CE, as the remaining rectangle AKB to the remaining rectangle DLE; and, by converſion, the ſquare of CB is to the rectangle AEB, as the rectangle AKB to its exceſs above the rectangle DLE, that is, as the rectangle AKB to [preceding prop.] the ſquare of KH; but [2. 2.] the rectangle AEB is equal to the ſquare of CF: as, therefore, the ſquare of CB to the ſquare of CF, ſo is the rectangle AKB to the ſquare of KH; the ſquare, therefore, of AB is to that of

FG, as the rectangle AKB to the ſquare of HK.

In the other caſe, let HP be the ſtraight Fig. 1. n. 1, 2. line drawn from the point parallel to the greater axis, and meeting the leſſer in P; the ſquare of FG is to the ſquare of AB, &c. The ſquare of CB, as hath been proved, is to the ſquare of CF, as the rectangle AKB to the ſquare of HK or PC; therefore the ſquare of CF [prop. B. and 19. 5. Elem.] is to the ſquare of CB, as the rectangle FPG to the ſquare of CK, that is, as the rectangle FPG to the ſquare of HP.

COR. 1. Hence the ſquares of ſtraight lines drawn from points of an ellipſe to either axis, ſo as to be parallel to the other axis, are to one another as the rectangles contained by the ſegments of the axis met, which are between the ſtraight lines and Fig. [illegible] its vertices. For let HM, PQ be parallel to the axis FG, and meet the other axis in M, Q; and, by the propoſition, the rectangle AMB is to the ſquare of HM as the ſquare of AB to the ſquare of FG, that is, as the rectangle AQB to the ſquare of PQ; and, alternately, the rectangle

AMB is to the rectangle AQB, as the ſquare of HM to the ſquare of PQ.

Cor. 2. If a circle be deſcribed upon either axis as a diameter, and MH, QP parallel to the other axis, meet it in the points N, R; the ſegments (of theſe parallels) between the axis, and the circumference of the circle, are to one another as the ſegments between the axis and the ellipſe: for the rectangles AMB, AQB are equal [35. 3. Elem.] to the ſquares of MN, QR, each to each: therefore the ſquare of MN is to the ſquare of QR, as the ſquare of MH to the ſquare of QP; conſequently the ſtraight lines MN, QR, MH, QP are alſo [22. 6. Elem.] proportionals. Fig. 2.

PROP. VII. THEOR.

Every ſtraight line terminated both ways in an ellipſe, and parallel to either axis, is biſected by the other axis; or, what is the ſame thing, the axes are conjugate diameters.

Let HL be parallel to the axis FG, and meet the other axis AB in M, and the circle deſcribed upon AB in the points N, O; then, as MN to MO, [2 cor. preceding prop.], ſo is MH to ML: and becauſe NO is cut at right angles by AB, MN is equal to [3. 3. Elem.] MO; therefore MH is equal to ML. Fig. 2.

PROP. VIII. THEOR.

Every ſtraight line terminated both ways in an ellipſe, and biſected by one axis, is parallel to the other.

HP biſected in S by the axis FG, is parallel to the other axis AB. Draw HM, PQ parallel to FC, and let them meet the circle deſcribed upon AB in the points N, R: then, becauſe HM, FC, PQ are parallels, as HS to SP, ſo is MC to CQ: but HS is equal to SP; therefore, alſo, MC is equal to CQ; conſequently MN is [14. 3. Elem.] equal to QR; and therefore HM is equal to [2 cor. 6. 2.] PQ: but HM is alſo parallel to PQ; therefore HP is [3. 3. 1. Elem.] parallel to MQ. Fig. 2.

Cor. Hence ſtraight lines HM, PQ, parallel to either axis FG, and which cut off between the centre and the points where they terminate in the other axis, equal ſegments MC, QC of this other axis, are equal to each other: and, on the contrary, if HM, PQ be equal to each other, and parallel to either axis, they cut off [2 cor. 6. 2.—14. 3. Elem.] equal ſegments MC, QC of the other axis.

PROP. IX. Theor.

Of all diameters the greater axis is the greateſt, and the leſs axis the leaſt; and a diameter which is nearer to the greater axis, is greater than one more remote.

Fig. 2. Let CB be half the greater axis, and FC half the leſs; let CH be any other ſemidiameter, and draw HM at right angles to CB: and becauſe the ſquare of CB is to the ſquare of CF as the rectangle AMB to the ſquare of HM, and that [4. cor. 1. 2.] CB is greater than CF; therefore the rectangle AMB is greater than the ſquare of HM;

HM: to theſe unequals add the common ſquare of CM; and the ſquare of CB [5. 2. Elem.] will be greater than that of CH [47. 1. Elem.]; and conſequently the ſtraight line CB will be greater than CH. Next draw HS at right angles to the leſs axis, and it may, in the ſame manner, be demonſtrated, that CF is leſs than CH; therefore, of all ſemidiameters, CB is the greateſt, and CF the leaſt.

Fig. 2. Let now CT be more remote from the greater axis than CH; CH is greater than CT: draw TV parallel to HM, and let it meet AB in V, and the ellipſe again in H, and let HZ be parallel to MV, and make CQ equal to CM.

Becauſe the rectangle AVB, which conſiſts of the * rectangles AMB and QVM, is

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* This is demonſtrated in prop. 31. book 7. of Pappus of Alexandria. The propoſition is to this purpoſe. "If in a ſtraight line AB two equal ſtraight "lines AQ, BM be taken, and betwixt Q and M a point V; the rectangle AVB is equal to the rectangles AMB, QVM together." For AB being biſected in C, the rectangle AVB, together with the ſquare of CV, is [5. 2. Elem.] equal to the ſquare of CB, that is, to the rectangle AMB, together with the ſquare of CM [5. 2. Elem.] that is, to the rectangle AMB, together with the

See fig. 2. of this book.

is to the ſquare of VT, which [5. 2. Elem.] is made up of the ſquare VZ and rectangle TZX, as [1. cor. 6. 2.] the rectangle AMB to the ſquare of MH or VZ; the whole the rectangle AVB, is to the whole the ſquare of VT, as [19. 5. Elem.] the remaining rectangle QVM to the remaining rectangle TZX: but the rectangle AVB is [35. 3. Elem.] greater than the ſquare of VT; therefore the rectangle QVM is alſo greater than TZX. To theſe unequals add the ſquare of CV, and the ſquare of CM [5. 2. Elem.] will be greater than the rectangle TZX together with the ſquare of CV. Superadd to the ſame unequals, the ſquare of MH or ZV, and the ſquares of CM, MH together, will be greater than the ſquares of VT, VC together; that is, the ſquare of CH is greater than the ſquare of CT; and therefore CH is greater than CT.

the rectangle QVM, and the ſquare of CV [5. 2. Elem.]. Take away the common ſquare of CV, and the remaining rectangle AVB is equal to both the remaining rectangles AMB, QVM.

LEMMA I.

If a point A be taken in a ſtraight line AE, and two parallels BC, DE be drawn on the ſame ſide of AE, and on the ſame ſide of the point A, or on the contrary ſides of AE, and on the contrary ſides of the point A; if theſe parallels have the ſame ratio to each other, as the ſegments of AE, which are intercepted between them and the point A; the extremities B, D of the parallels and the point A are in the ſame ſtraight line. Fig. 3. 4.

In the one caſe, complete the parallelograms AB, AD; theſe are ſimilar, and have the angle at A common: conſequently they are about the ſame [26. 6. Elem.] diameter, that is, the points A, B, D are in the ſame ſtraight line. Fig. 3.

In the other caſe, join BA, DA: and becauſe BC is to CA as DE to EA, and the Fig. 4.

angle

angle BCA equal to DEA, the triangles ABC, ADE are equiangular, [6. 6. Elem.]; conſequently the angle EAD is equal to the angle CAB: and therefore AB, AD are [14. 1. Elem.] in the ſame ſtraight line.

LEMMA II.

Fig. 5. 6. 7. 8. If two parallel ſtraight lines AC, BD be drawn from two points A, B, and other two parallels AE, BF, either coinciding or not coinciding with the parallels AC, BD, and having the ſame ratio to each other as the parallels AC, BD, and lying on the ſame ſide of AB, or on the contrary ſides of it, according as the ſame parallels AC, BD are on the ſame ſide of AB or on the contrary ſides of it, be placed from the ſame points; if a ſtraight line be drawn through the extremities of either of the pairs of parallels, the point where it meets AB, and the extremities of the other two parallels, are in the ſame ſtraight line.

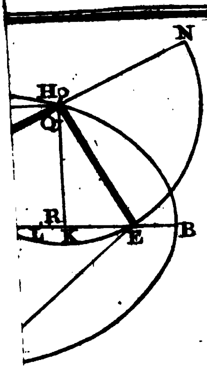
Let a ſtraight line be drawn, for example, through C, D the extremities of AC, BD, and let it meet the ſtraight line AB in the point G; the extremities E, F of the other two parallels AE, BF, and the point G, are in the ſame ſtraight line. For ſince the triangles AGC, BGD are equiangular, AG is to BG as AC to BD: but AC is to BD (by hypotheſis) as AE to BF; therefore AG is to BG as AE to BF: conſequently, by the preceding lemma, the points G, E, F are in one and the ſame ſtraight line. In like manner, if a ſtraight line be drawn through the extremities E, F of the parallels AE, BF, the point where it meets AB, and the extremities C, D of the other two parallels, are in the ſame ſtraight line.

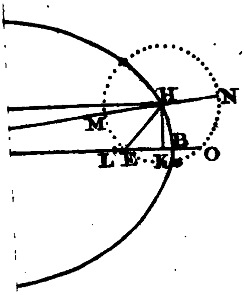
LEMMA III.

The ſame things being ſtill ſuppoſed which are ſuppoſed in the ſecond lemma, if through the extremities of either of the pairs of parallels two ſtraight lines be drawn parallel to each other, and meeting AB; the ſtraight lines that join the two points where AB is met, and the correſponding extremities of the other two parallels, are likewiſe parallel.

Fig. 9. 10. 11. For example, let CH, DK be drawn parallel to each other through the extremities CD of the parallels AC, BD; and let them meet A, B in the points H, K; the ſtraight lines EH, FK, which join the extremities of the other two parallels AE, BF, and the correſponding points H, K in the ſtraight line AB, are likewiſe parallel.

Becauſe both AC, BD, and CH, DK, are parallels, the triangles ACH, BDK are equiangular; therefore AH is to BK as





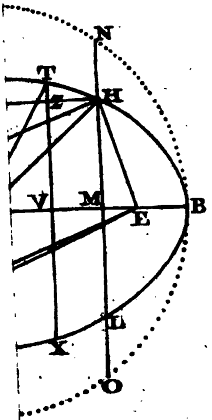


Fig. 3.

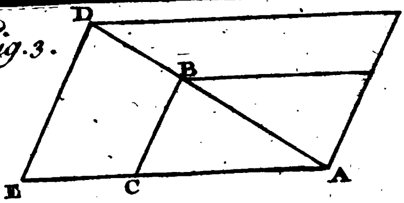


Fig. 4.

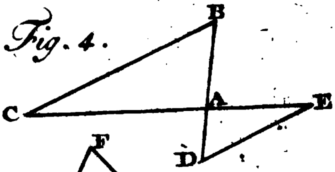


Fig. 5.

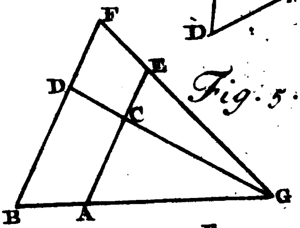


Fig. 6.

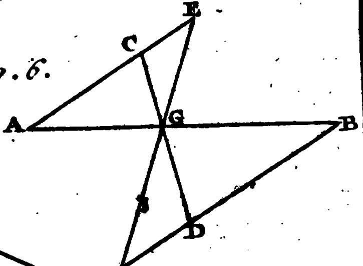


Fig. 7.

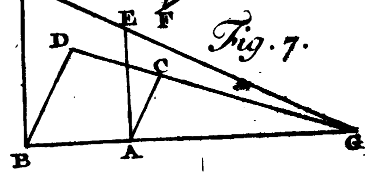
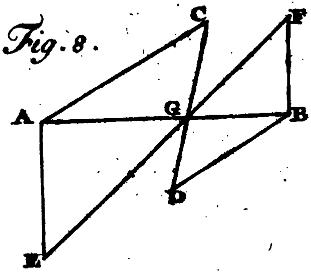


Fig. 8.



ag. 34.

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as AC to BD, that is, by hypotheſis, as AE to BF; and the angle EAH is equal to FBK: therefore the triangle AHE [6. 6. Elem.] is equiangular to BKF: therefore the angle AHE is equal to BKF; and conſequently EH is [27. or 28. 1. Elem.] parallel to FK.

PROP. X. Theor.

If from any point (the vertex of either of the axis excepted) of an ellipſe, a ſtraight line be drawn parallel to either axis, and meeting a circle deſcribed upon the other axis, and another ſtraight line be drawn touching the circle in the point where the parallel meets it; this other ſtraight line meets the common diameter of the ellipſe and the circle; and a ſtraight line drawn from the point where the common diameter is met, to the point in the ellipſe, touches the ellipſe: alſo a ſtraight line which is drawn through the vertex of either axis parallel to the other axis, touches the ellipſe.

Fig. 12. Let E be the point in the ellipſe, and EF a ſtraight line drawn parallel to either axis CD, and meeting a circle deſcribed upon AB in G; let GN touch the circle in G, and meet the common diameter AB in H; the ſtraight line that joins the point H and the point taken in the ellipſe touches the ellipſe.

Let C be the centre of the ellipſe and circle, and join CG; then becauſe CGN is a right angle, and GCB leſs than the right angle DCB, GN, CB will neceſſarily meet: let them meet in H, and join HE; HE will touch the ellipſe: for if not, let it, if poſſible, meet the ellipſe again in K, and through K draw a ſtraight line parallel to EF, and let the parallel through K meet AB in L, and the circle in M: then, becauſe EF is to KL as GF to ML, [2. cor. 6. 2.] and that the points H, K, E are in one ſtraight line; the points H, M, G are likewiſe in one [lemma 2.] ſtraight line: the ſtraight line, therefore, HG meets the circle in two points G and M: but, by hypotheſis, the ſame HG touches the circle; which is abſurd: HE, therefore, meets not the ellipſe any where but in the point E; and therefore it touches it in this point.

Next, through D, the vertex of either axis CD, let a ſtraight line be drawn parallel to the other axis AB; this ſtraight line touches the ellipſe: for if not, let it, if poſſible, meet the ellipſe again in O; and through O draw a ſtraight line parallel to CD; and let the parallel through O meet AB in P, and the circle in Q; alſo let CD meet the circle in R: then, becauſe CO is a parallelogram, CD is equal to PO: but as CD to PO, ſo is CR to PQ, [2. cor. 6. 2.]; CR, therefore, and PQ are equal: but they are alſo parallel; therefore, if QR be joined, it will be parallel to CP; and the angle QRC is, therefore, a right angle. And thus RQ, drawn perpendicular to a diameter of the circle from the extremity of that diameter, falls within the circle; which is abſurd, [16. 3. Elem.]; therefore DO touches the ellipſe. Fig. 12.

Cor. 1. If EH touch the ellipſe, and GH be joined, it may be ſhewn in the ſame manner that GH touches the circle.

Cor. 2. From a point in an ellipſe, only one ſtraight line can be drawn which will touch that ellipſe: were it poſſible for two ſtraight lines to touch the ellipſe in one and the ſame point, two could alſo touch the circle in the ſame point; and thus the cor. to 16. 3. Elem. would be falſe.

Cor. 3. A ſtraight line cannot meet an ellipſe in more than two points; for could a ſtraight line meet the ellipſe in more than two points, a ſtraight line could alſo meet the circle in more than two points; which would be an abſurdity: The ellipſe, therefore, is convex on the ſide where ſtraight lines touch it, but concave on the contrary ſide.

Cor. 4. An angle contained by any diameter, (the axis excepted), and a ſtraight line that touches the ellipſe in the vertex of that diameter, and is drawn towards the greater axis, is greater than a right angle. Let CE be the diameter, EH the tangent, and AB the greater axis, the other things remaining as in the propoſition ; the angle CEH is greater than CGH, that is, it is [21. 1. Elem.] greater than a right angle.

COR. 5. The propoſition points out a method by which, if the greater axis be given in poſition and magnitude, a ſtraight line can be drawn that will touch an ellipſe in a given point.

COR. 6. Two ſtraight lines touching an ellipſe in the vertices of a diameter are parallel.

Let EH, ST touch the ellipſe in the vertices of the diameter ECS, and let them meet the axis AB in H, T; draw EF, SV perpendicular to the ſame axis, and let EF, SV meet the circle deſcribed upon it in G, X; join GH, XT, which, by cor. 1. will touch the circle: and becauſe GF, XV are divided in the ſame ratio, [2. cor. 6. 2.] in the points E, S, and that ECS is a ſtraight line; therefore the points G, C, X are alſo in a ſtraight line [lem. 2.]: and becauſe GH, XT, which touch the circle in the vertices of the diameter GX, are parallels, EH, ST are alſo parallels, [lem. 3.] Fig. 12.

PROP. XI. Theor.

If from a point of an ellipſe two ſtraight lines be drawn to the foci, and a ſtraight line be drawn biſecting the angle adjacent to that contained by thoſe two ſtraight lines; that ſtraight line touches the ellipſe.

Fig. 13. Caſe 1. When the ſtraight line biſecting the angle is parallel to the greater axis, let AB be the greater axis, C the centre, and D, E the foci; from a point F of the ellipſe draw the ſtraight lines FD, FE; let FH, which biſects the angle EFG, be parallel to the greater axis AB; FH touches the ellipſe: making FG equal to FE, join EG, and let it meet FH in H; then, ſince FE is equal to FG, and FH common, and the angle EFH equal to GFH, EH is equal to HG: and FH, ED being parallels, EH is to HG as DF to FG; DF, therefore, is equal to FG, that is to FE; and in the triangles DFC, EFC, FC is common, and DC equal to CE; therefore the angles at

C are equal: and thus each of them is a right angle; therefore CF is [def. 6. 2.] the half of the leſs axis; and, conſequently, FH, which is parallel to the other axis, touches [10. 2.] the ellipſe.

Caſe 2. The conſtruction remaining the ſame in other reſpects as in the preceding caſe, let now FH meet the greater axis in K, and draw FL at right angles to the ſame axis, and let it meet the circle deſcribed upon AB in the point M: joining MK, and drawing MC to the centre, make BN equal to EF; AN will conſequently [1. 2.] be equal to DF: and ſince the outward angle EFG of the triangle DFE is divided into two equal parts by the ſtraight line FK, which meets the baſe DE in the point K, DK is to KE as DF to FE [A. 6. Elem.], that is, as AN to NB: and, by compoſition, DK together with KE is to KE as AB to NB: take the halves of the antecedents, and CK will be to KE as CB to BN; and, by converſion and alternation, CK is to CB as CE to CN, that is, as CB to CL [4. 2.]: but CB is equal to CM; therefore, as CK to CM, ſo is CM to CL: therefore the triangle CMK is equiangular [6. 6 Elem.] to CLM: but CLM is a right angle; Fig. 14.

therefore the angle CMK is alſo a right angle; and conſequently the ſtraight line MK touches the circle, [16. 3. Elem.]; therefore FK touches the ellipſe.

Otherwiſe: produce DF to G, ſo that FE may be equal to FG; and in FK take any point O, and join OD, OE, OG, and EG, and let this laſt meet FK in P; then, becauſe FE is equal to FG, FP common, and the angle EFP equal to GFP; EP, GP are equal, and the angles at P right angles; therefore OE is equal to OG: but DO, OG are together greater than DG; conſequently DO, OE are together greater than the ſame DG, that is, than DF, FE together, that is, than the greater axis AB: the point O is, conſequently, without the ellipſe, [3. cor. 1. 2.]; and, conſequently, FK touches the ellipſe.

Cor. On the contrary, if a ſtraight line FK touch the ellipſe, and DF, FE be drawn from the point of contact to the foci; the angles DFO, EFK, which DF, FE make with the touching line in contrary directions of it, are equal. For let them be unequal; then DF being produced to G, the angles EFK, GFK are likewiſe unequal:

but by the propoſition, a ſtraight line which biſects the angle EFG touches the ellipſe; and by the hypotheſis, FK touches it in the point F; which is [2. cor. 10. 2.] abſurd.

PROP. XII. PROB.

The greater axis of an ellipſe being given in poſition and magnitude, and the foci being given, to draw a ſtraight line parallel to another ſtraight line given in poſition, and which ſhall touch the ellipſe.

Let AB be the greater axis, D, E the foci; let ST be given in poſition: from E, either of the foci, draw EV at right angles to ST, and from the centre D, with a diſtance equal to AB, deſcribe a circle: this circle will meet EV in two points; of which let G be either; and having joined DG, draw EF to it, and let EF make the angle GEF equal to EGD; and through F draw FK parallel to ST; FK touches the ellipſe in the point F. Fig. 14.

Becauſe the angle GEF is equal to EGF, EF is equal to FG; therefore DF, FE are together equal to DG, that is, to the greater axis AB: the point F is, therefore, in [3. cor. 1. 2.] the ellipſe. Let FK meet the ſtraight line VE in P; and becauſe FK, ST are parallels, EV at right angles to ST is likewiſe at right angles to FP: therefore the angles EFP, GFP are equal; and therefore FK touches the ellipſe. In like manner, by employing the other point, where the circle deſcribed from the centre D meets EV, another tangent can be drawn parallel to ST.

PROP. XIII. THEOR.

Every ſtraight line parallel to a ſtraight line that touches the ellipſe, and terminated both ways by the ellipſe, is biſected by the diameter that paſſes through the point of contact.

If the diameter be either of the axes, a tangent drawn through its vertex is parallel to the other axis, [10. and 2. cor. 10. 2.];

and thus a ſtraight line parallel to the tangent is alſo parallel to this other axis; and conſequently is biſected by the diameter (or axis) that paſſes through the point of contact [7. 2.].

But let CE be any other diameter, and in its vertex let EF touch the ellipſe; let GH, parallel to EF, be terminated in G, H, two points of the ellipſe; and let GH meet the diameter CE in K; GK is equal to KH.

Fig. 15. n. 1, 2.

Let the tangent EF meet either axis AB in F; let GH meet the ſame axis in L; through the points E, G, H draw EM, GN, HO perpendiculars to AB, and let them meet the circle deſcribed upon AB in the points P, Q, R; join FP, LQ, CP; let LQ meet CP in the point S, and join SK: then, becauſe EF touches the ellipſe, FP touches the circle; and becauſe the parallels QN, RO are cut in the ſame ratio in the points G, H, and that HGL is a ſtraight line, the points L, Q, R are in a ſtraight line [lemma 2.]. Again, ſince the parallels PM, QN are cut in the ſame ratio in the points E, G, through which the parallels EF, GL are drawn, FP, LQ are likewiſe parallels [lem. 3.]; CS, conſequently,

ſequently, is to SP as CL to LF, that is, as CK to KE: therefore KS is [2. 6. Elem.] parallel to EP, and conſequently to QN, RO; therefore GK is to KH as QS to SR: but QS is equal to SR, becauſe CP being at right angles to PF, which touches the circle, is alſo at right angles to RQ, which is parallel to PF; therefore GK is equal to KH.

Cor. 1. On the contrary, any ſtraight line GH, terminated both ways by the ellipſe, and biſected by the diameter CE; or, in other words, any ſtraight line ordinately applied to the diameter CE, is parallel to a ſtraight line EF, which touches the ellipſe in the vertex of this diameter. For, if not, draw a tangent that ſhall be [12. 2.] parallel to GH; and GH will be biſected by the diameter which paſſes through the point of contact: but, by hypotheſis, the ſame GH is biſected by another diameter CE; which is abſurd.

Cor. 2. All ſtraight lines ordinately applied to the ſame diameter are parallel to one another.

Cor. 3. If two or more parallels be terminated both ways by an ellipſe, the dia-

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meter which biſects the one, or one of them, biſects alſo the other, or the reſt of them: for that one which is biſected by a diameter, is parallel to the ſtraight line touching the ellipſe in the vertex of that diameter; and conſequently the other, or the reſt, is or are parallel to the ſame tangent; and therefore is or are biſected by the ſame diameter.

COR. 4. But a ſtraight line which biſects two parallels terminated both ways by an ellipſe, is a diameter: for if it be not, draw a diameter biſecting the one of the parallels, and this diameter will alſo biſect the other of them: but, by hypotheſis, each of them is biſected by another ſtraight line; which is abſurd. And if from the point of contact, a ſtraight line be drawn biſecting another ſtraight line parallel to a ſtraight line that touches the ellipſe, and terminated both ways by the ellipſe, that ſtraight line is a diameter: for if it be not, draw a diameter through the point of contact; this diameter biſects the parallel to the tangent; which is abſurd.

COR. 5. A ſtraight line which is drawn through the vertex of a diameter, and is parallel to an ordinate to that diameter, touches the ellipſe.

Cor. 6. Two ſtraight lines in an ellipſe, which paſs not through the centre, do not biſect each other: for if they biſected each other, they would be each of them parallel to a ſtraight line touching the ellipſe in the vertex of the diameter drawn through the point of biſection, and of conſequence they would be parallel to each other; which is abſurd.

PROP. XIV. Theor.

Two diameters of an ellipſe, either of which is parallel to a ſtraight line touching the ellipſe in either vertex of the other, are conjugate diameters.

Fig. 15. n. 1. 2. Let ET, VX be two diameters; let either of them, VX, be parallel to EF, a ſtraight line touching the ellipſe in the vertex E of the other; ET, VX are conjugate diameters.

Through the vertices E, V draw to either axis AB ſtraight lines EM, VY parallel to the other axis CD; and let EM, VY meet the circle deſcribed upon AB in the points P, Z, on the ſame ſides of the circle on which are the points E, V; and from the point F, where EF meets the axis AB, draw the ſtraight line FP, which will touch the [1. cor. 10. 2.] circle in P; and from the point Z, the ſtraight line Z*a* touching the circle, and meeting the axis in *a*; join *a*V; *a*V will touch the [10. 2.] ellipſe; and, laſt of all, from the centre draw CP, CZ.

Then, becauſe the parallels PM, ZY are cut in the ſame ratio in the points E, V through which are drawn the parallels EF, VC; FP, CZ are [lem. 3.] alſo parallels: but CPF is a right angle; conſequently PCZ is alſo a right angle: and CZ*a* is a right angle; therefore CP, Z*a* are parallels: and conſequently CE and *a*V are parallels: therefore ſtraight lines parallel to CE will likewiſe be parallel to V*a*, which touches the ellipſe in V; and of conſequence they will be biſected [13. 2.] by the diameter CV: and becauſe CV, by hypotheſis, is parallel to EF, all ſtraight lines parallel to CV will likewiſe be parallel to EF, and will conſequently be biſected by

the

the diameter CE; therefore CE, CV are conjugate diameters [7. def. 2.]

Cor. 1. And ſince CV is the only diameter that can biſect ſtraight lines parallel to CE, and terminated both ways by the ellipſe, CV alone is conjugated to CE.

Cor. 2. If, therefore, CE, CV be conjugate diameters, each of them is parallel to the tangent drawn through the vertex of the other.

Cor. 3. On the other hand, a ſtraight line which, when drawn through the vertex of a diameter, is parallel to the diameter conjugated to that diameter, touches the ellipſe in that vertex.

Cor. 4. A ſtraight line parallel to a diameter, and terminated both ways by the ellipſe, is biſected by the diameter conjugated to that diameter: for it is parallel to the ſtraight line which touches the ellipſe in the vertex of this diameter conjugated to the other; and conſequently it is biſected by this ſame conjugated diameter. On the other hand, a ſtraight line biſected by a diameter, is parallel to the diameter conjugated to that diameter.

PROP. XV. THEOR.

If from a point in an ellipſe to either of two conjugate diameters, a ſtraight line be drawn parallel to the other, the ſquare of the diameter to which the ſtraight line is drawn, is to the ſquare of the other diameter, as the rectangle contained by the ſegments into which the ſtraight line divides the diameter to which it is drawn, is to the ſquare of that ſtraight line.

Let ET, VX be conjugate diameters, and from a point G of the ellipſe draw GK to ET, parallel to VX; the ſquare of ET is to the ſquare of VX as the rectangle EKT to the ſquare of GK. Fig. 15. n. 1. 2.

For ſince ET, VX are conjugate diameters, VX is parallel to the tangent EF, drawn through the vertex E of ET: and the ſame conſtruction as in the two foregoing propoſitions ſtill remaining, and what was there demonſtrated being ſtill kept in view, let SK, when produced, meet AB in the point β; and through G draw the ſtraight line Gγ parallel to the ſame AB, and meeting Sβ in γ; then, becauſe PCZ is a right angle, and that the angles CPM, MCP are together equal to a right angle; the angle PCZ is equal to the ſame CPM and MCP together: take away the common angle MCP, and the remaining angle MCZ is equal to the remaining CPM; and CP, CZ are equal; therefore the right-angled triangles PMC, CYZ are equal; PM is, therefore, equal to CY: but ſince PM, Sβ, and PF, SL are parallels, the triangles PFM, SLβ are equiangular: and thus CPM, SLβ are [8. 6. Elem.] alſo equiangular; therefore CP is to PM as SL to Lβ, that is, as SQ to Nβ [2. 6. Elem.]; alternately CP is to SQ as PM to Nβ: but PM having been proved equal to CY, and that Gγ is equal to Nβ; PM is to Nβ as CY to Gγ: and the triangles CVY, GKγ being equiangular, CY is to Gγ as CV to GK; therefore CP is to SQ as CV to GK: hence the ſquare of CP is to the ſquare of SQ, as the ſquare of CV to the ſquare of GK: but the ſquare of CP is to the ſquare of CS as the ſquare of CE to the ſquare of CK; and, by converſion, and prop. 47. 1. Elem. the ſquare of CP is to the ſquare of SQ as the ſquare of CE to the rectangle EKT: the ſquare, therefore, of CE is to the rectangle EKT, as the ſquare of CV to that of GK; alternately, the ſquare of CE is to the ſquare of CV as the rectangle EKT to the ſquare of GK: therefore the ſquare of ET is to that of VX, as the rectangle EKT to the ſquare of GK.

Univerſally, then, the ſquare of any diameter is to the ſquare of its conjugate, as the rectangle contained by the ſegments, intercepted between its vertices and a ſtraight line ordinately applied to it, is to the ſquare of the ſegment (intercepted between the ellipſe and it) of the ſtraight line ordinately applied: for an ordinate to a diameter is parallel to the tangent drawn through the vertex of that diameter; and therefore is parallel to the diameter conjugated to that diameter.

COR. 1. The ſquares of ſtraight lines ordinately applied to the ſame diameter, are to one another as the rectangles contained by the ſegments of that diameter, as was demonſtrated [1. cor. 6. 2.] with regard to the axes.

Cor. 2. If ET, VX be conjugate diameters of an ellipſe AT, and from a point G a ſtraight line GK be drawn parallel to VX, one of the conjugates, and meeting the other ET in K; and if the ſquare of ET be to the ſquare of VX as the rectangle EKT to the ſquare of GK; the point G is in the ellipſe: for if GK meet not the ellipſe in the point G, on that ſide, on which G is, of the diameter ET; let it meet the ellipſe in δ: then, by the propoſition, the rectangle EKT is to the ſquare of δK as the ſquare of ET to the ſquare of VX, that is, by hypotheſis, as the rectangle EKT to the ſquare of GK: hence the ſquare of δK is equal to the ſquare of GK; and thus the ſtraight line δK is equal to the ſtraight line GK; which is impoſſible.

Cor. 3. If from two points G, ε, one o which, ε, is in the ellipſe, there be draw to the diameter ET ſtraight lines GK, parallel to ſtraight lines ordinately appli to the ſame ET; if the rectangles EKT, EεT contained by the ſegments of the diamet ET, which are intercepted between its ve tices and the parallels drawn to it, have the ſame ratio to each other as the ſquares of GK, κζ; the other point G is likewiſe in the ellipſe. This is demonſtrated from cor. 1. in the ſame manner as the ſecond corollary from the propoſition.

Fig. 16

COR. 4. If a circle be deſcribed upon AB, a diameter of the ellipſe, and ſtraight lines DE, FG be drawn ordinately applied to the diameter AB; and from the points D, F ſtraight lines DH, FK be drawn perpendicular to the ſame AB, and meeting the circle in the points H, K; the perpendiculars DH, FK have the ſame ratio to each other which the ordinates DE, FG have; for the ſquares of DE, FG have the ſame ratio to each other which the rectangles ADB, AFB have, that is, which the ſquares [35. 3. Elem.] of DH, FK have: therefore DH is to FK as DE to FG, [22. 6. Elem.]

COR. 5. And if two ſtraight lines ordinately applied to a diameter, cut off, between the centre and the points where they meet that diameter, equal ſegments of that diameter, they are equal: and if equal, they cut off, between the centre and theſe points, equal ſegments.

PROP. XVI. THEOR.

If from a point of an ellipſe a ſtraight line be ordinately applied to a diameter, and from the point where the ordinate meets the diameter a perpendicular be drawn meeting the circumference of a circle deſcribed upon the diameter; if a ſtraight line touching the circle in the extremity of the perpendicular meet the diameter; the ſtraight line that joins the point where that tangent and the diameter meet, and the point in the ellipſe, touches the ellipſe. On the other hand, if the ſtraight line touching the ellipſe meet the diameter, the ſtraight line that joins the point where they meet, and the extremity of the perpendicular, touches the circle in that extremity.

E being a point in the ellipſe, ED drawn ordinately applied to AB a diameter, and there being drawn from the point D, where the ordinate and diameter meet, a perpendicular DH, that meets the circle deſcribed upon AB in H; if a ſtraight line touching the circle in H meet the diameter AB in the point L; the ſtraight line EL that joins L and the point E in the ellipſe, touches the ellipſe in the point E. Fig. 16.

For if EL touch not the ellipſe, let it, if poſſible, meet it in another point M; and through M to the diameter AB draw MN parallel to ED; and let NO, drawn through N at right angles to the diameter AB, meet the circle in O on the ſame ſide of AB on which H is: ſince, then, the parallels DE, NM are drawn from the points D, N, as alſo the parallels DH, NO, which have the ſame ratio to each other [4 cor. preced.] which DE, NM have, and that the points E, M, L are in a ſtraight line; therefore the points H, O, L [lem. 2.] are in a ſtraight line; the ſtraight line LH, therefore, cuts the circle: but, according to the hypotheſis, it touches the circle; which is abſurd: therefore LE touches the ellipſe.

It may be ſhewn, after the ſame manner, that if EL touches the ellipſe, LH touches the circle.

Cor. Hence a method is pointed out, by which if a diameter of an ellipſe be given in poſition and magnitude, and the angle which that diameter makes with any ſtraight line ordinately applied to it, be given, a ſtraight line can be drawn that will touch the ellipſe in a given point.

PROP. XVII. Theor.

If from a point of an ellipſe a ſtraight line be drawn touching that ellipſe, and meeting a diameter; if from the point of contact a ſtraight line be ordinately applied to that diameter; the ſemidiameter is a mean proportional between the ſegments of the diameter that are intercepted between the centre and the tangent, and between the centre and the ordinate; and the ſegments (of the diameter) intercepted between its vertices and the tangent, have the ſame ratio to each other as the ſegments (of the ſame diameter) intercepted between its vertices and the ordinate.

E being a point in the ellipſe, EL touching the ellipſe, and meeting the diameter AB in L, and there being drawn from the point of contact a ſtraight line ED ordinately applied to the diameter AB; the ſemidiameter CB is a mean proportional between CL, CD, the ſegments (of the diameter) intercepted between the centre and the tangent, and between the centre and the ordinate. Fig. 16.

Having deſcribed a circle upon the diameter AB, and drawn from the point D a ſtraight line DH at right angles to AB, and meeting the circle in H, join HL: then, becauſe LH touches the [16. 2.] circle, and that HD is perpendicular to the diameter, CD, CB, CL are proportionals, [8. 6. Elem.]

Next, becauſe CL is to CB as CB to CD,

by converſion CL is to BL as CB to BD: double the antecedents, and twice CL is to BL as AB to BD; and, by diviſion, AL is to BL as AD to BD.

Cor. 1. Hence the rectangle contained by the ſegments (of the diameter AB) intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to that contained by the ſegments between the ordinate and the vertices of the diameter: for as CD, CB, CL are proportionals, the ſquare of CB is equal to the rectangle DCL; but the ſquare of CB is equal to the rectangle ADB, together with the ſquare of CD, [5. 2. Elem.]; and the rectangle DCL is equal to the rectangle CDL, together with the ſame ſquare of CD, [3. 2. Elem.]: therefore the rectangle ADB, and the ſquare of CD, are together equal to the rectangle CDL added to the ſame ſquare of CD: take away the common ſquare of CD, and there remains the rectangle ADB equal to CDL.

Cor. 2. And the rectangle contained by the ſegments (of the diameter AB) intercepted between the tangent and the centre, and between the tangent and the ordinate, is equal to that contained by the ſegments between the tangent and the vertices of the ſame diameter AB: for the rectangle ALB and ſquare of CB are together equal [6. 2. Elem.] to the ſquare of CL; and the rectangles LCD, CLD are together equal [2. 2. Elem.] to the ſame ſquare of CL: therefore, becauſe the ſquare of CB has been proved equal to the rectangle DCL, there remains the rectangle ALB equal to the rectangle CLD.

PROP. XVIII. THEOR.

From a point of an ellipſe let a ſtraight line be ordinately applied to a diameter, and from the ſame point let a ſtraight line be drawn meeting the diameter on the ſame ſide (of the ordinate) on which the nearer (to the ordinate) of the vertices of the diameter is ſituated: if the ſegment (of the diameter) intercepted between the centre and the point where the diameter and the ſtraight line meet, and the ſemidiameter, and the ſegment between the centre and the ordinate be proportionals; if, next, the ſegments between the point of concourſe of the diameter and ſtraight line, and the vertices of the diameter, and between the ordinate and thoſe vertices, be proportionals; if, thirdly, the ſegments between the ordinate and point of concourſe of the diameter and ſtraight line, between the ordinate and vertices of the diameter, and between the ordinate and centre, be proportionals; or, laſt of all, if the ſegments between the point of concourſe of the diameter and ſtraight line, and the more remote (from the ordinate) of the vertices of the diameter; between the ſame point of concourſe and the centre, between the ſame point of concourſe and the ordinate, and between the ſame point of concourſe and the other vertex of the diameter, be proportionals: in all theſe caſes, the ſtraight line drawn from the point in the ellipſe to the diameter produced, touches the ellipſe.

Let E be the point in the ellipſe, and ED an ordinate to the diameter AB; and let EL, drawn from the point in the ellipſe, meet the diameter in L: if the ſegment (of the diameter) CL, intercepted between the centre and point of concourſe L of the diameter and EL, the ſemidiameter CB, and the ſegment DC, between the ordinate and centre, be proportionals; next, if the ſegments AL, BL, between the point L and the vertices of the diameter, have the ſame ratio to each other, as the ſegments AD, BD, between the ordinate and Fig. 16.

vertices; if, thirdly, the ſegments DL, DA, DB, DC, between the ordinate and each of the points L, A, B, C, be proportionals; or, laſt of all, if the ſegments AL, CL, DL, BL, between the point L and each of the points A, C, D, B be proportionals: in all theſe caſes the ſtraight line EL touches the ellipſe.

Caſe 1. If EL does not touch the ellipſe, let EP touch it: therefore, by the preceding propoſition, CP, CB, CD are proportionals: but, by hypotheſis, CL, CB, CD are likewiſe proportionals; CP is, therefore, equal to CL; which is abſurd: conſequently EL touches the ellipſe.

Caſe 2. Becauſe, by hypotheſis, AL is to BL as AD to BD, by compoſition AL and BL together are to BL as AB to BD: take the halves of the antecedents, and CL is to BL as CB to BD; and, by converſion, CL is to CB as CB to CD: therefore, by the firſt caſe, EL touches the ellipſe.

Caſe 3. Since DL is to DA as DB to DC, by compoſition LA is to AD as BC or CA to CD; the remainder CL is, therefore, to the remainder AC or CB, as CB to CD; and, therefore, by the firſt caſe, again, EL touches the ellipſe.

Caſe 4. By hypotheſis, AL is to CL as DL to BL; therefore, by diviſion, AC or CB is to CL as DB to BL; CB, therefore, is to CL as the remainder CD to the remainder CB; and, inverſely, CL is to CB as CB to CD: therefore EL touches the ellipſe.

PROP. XIX. THEOR.

If from two vertices of two conjugate diameters two ſtraight lines be ordinately applied to another diameter, the ſquare of the ſegment (of that other diameter) intercepted between either ordinate and the centre, is equal to the rectangle contained by the ſegments between the other ordinate and the vertices of that ſame diameter.

Let CA, CB be the two conjugate diameters, of which the points A, B are vertices; from A, B let AF, BG be ordinately applied to another diameter DE; the ſquare Fig. 171

ſquare of CG, intercepted between the ordinate BG and the centre, is equal to the rectangle EFD contained by the ſegments intercepted between the other ordinate AF and the vertices of DE; and the ſquare of CF, in like manner, is equal to the rectangle EGD.

Draw AH, BK touching the ellipſe in A, B, and meeting the diameter ED in H, K: then, becauſe both CB, AH, and BG, AF, are parallel, the triangle CBG is equiangular to HAF; and becauſe BK is parallel to CA, the triangle GBK is alſo equiangular to HAC; therefore CG is to FH as CB to AH, that is, as CK to CH: but CD is a mean proportional both between CG, CK, and between CF, [16. 2.] CH; therefore CF is to CG as CK to CH: and CG, as hath been ſhewn, is to FH as CK to CH; therefore as CF to CG, ſo is CG to FH: and, conſequently, the ſquare of CG is equal to the rectangle CFH: but the rectangle EFD is equal to the ſame [1. cor. 17. 2.] CFH; hence the ſquare of CG is equal to the rectangle EFD; and take theſe equals from the ſquare of CD, and there will remain the rectangle EGD equal to the ſquare of CF, [ſee prop. 5. 2. Elem.]

COR. 1. Hence the ſemidiameter CD, to which the ordinates are drawn, is to the ſemidiameter conjugated to it as the diſtance between either ordinate and the centre is to the other ordinate: for the ſquare of CD is to the ſquare of CL as the rectangle EFD to the ſquare of AF, that is, (by the propoſition), as the ſquare of CG to the ſquare of AF; therefore CD is to CL as CG to AF. In like manner, it may be ſhewn, that CD is to CL as CF to BG.

COR. 2. The ſquares of the ſegments of the diameter to which the ordinates are drawn, which ſegments are between the ordinates and the centre, are together equal to the ſquare of the ſemidiameter: for ſince the ſquare of CG is equal to the rectangle EFD, the ſquare of CF, together with the ſquare of CG, is equal to the ſquare of CF, together with the rectangle EFD, that is, to the [5. 2. Elem.] ſquare of CD.

COR. 3. Hence the ſum of the ſquares of any two conjugate diameters is equal to the ſum of the ſquares of the axes. Let CD, CL be the ſemiaxes, and CA, CB conjugate ſemidiameters; let AF, BG be perpendiculars to CD, and AM, BN perpendiculars to CL: then, becauſe the ſquare of CD, as was proved in the preceding corollary, is equal to the ſquare of CF, together with the ſquare of CG, and that, by the ſame corollary, the ſquare of CL is equal to the ſquare of CM, together with that of CN, that is, to the ſquare of AF, together with that of BG; the ſquares of CD, CL are together equal to the ſquares of CF, CG, AF, BG: but the ſquares of AC, BC are together (47. 1. Elem.) equal to the ſame ſquares of CF, CG, AF, BG; therefore the ſquares of AC, BC are together equal to the ſquare of CD added to the ſquare of CL: Hence the ſum of the ſquares of two conjugate diameters is equal to the ſum of the ſquares of the axes.

PROP. XX. THEOR.

If through the vertices of two conjugate diameters four ſtraight lines be drawn touching the ellipſe; the parallelogram formed by theſe ſtraight lines, is equal to that formed by the tangents drawn through the vertices of

any other two conjugate diameters.

Let the ſtraight lines OV, OY, YX, XV touch the ellipſe in the vertices A, B, and in the vertices oppoſite to A, B of two conjugate diameters AC, BC; in like manner, let PT, PR, RS, ST touch the ellipſe in the vertices of the conjugate diameters DC, LC; the figures OVXY, PRST are [6. cor. 10. 2.] parallelograms, and equal to each other. Fig. 17.

To the diameter CD draw AF, BG parallel to CL, and to the diameter CL draw AM, BN parallel to CD; and let AO, BO meet the ſame CD in the points H, K; and having joined BH, complete the parallelogram HCNQ:

Then, becauſe AH touches the ellipſe, and that AF is drawn ordinately applied to the diameter CD, CH is to CD as CD to CF: and, by the firſt corollary of the foregoing propoſition, CD is to CL as CF to BG: therefore, by an inference, founded on what is called *equality of diſtance*, [ſee 19. def. B. 5. Elem.] CH is to CL as CD to BG or QH; and the angles DCL, CHQ are equal, for they are alternate; therefore the parallelogram DL is equal to the parallelogram [14. 6. Elem.] NH; but NH is the double of the triangle CBH, upon the ſame baſe, and between the ſame parallels; and the parallelogram ACBO is the double of the ſame triangle CBH, upon the ſame baſe CB, and between the ſame parallels CB, AH; ACBO, therefore, is equal to NH; and the parallelogram DL, as hath been ſhewn, is equal to the ſame NH; therefore the parallelograms DL, AB are equal; and thus the parallelograms PRST, OVXY, which are the quadruples of DL, AB, are equal.

PROP. XXI. THEOR.

If a ſtraight line touching an ellipſe meet two conjugate diameters, the rectangle contained by its ſegments, between the point of contact and the diameters, is equal to the ſquare of the ſemi-diameter conjugated to tha which paſſes through the poin of contact.

Let the ſtraight line HZ touch the ellipſe in the point A, and let it meet the conjugate diameters CD, CL in H, Z; let the ſemidiameter CB be conjugated to CA; the rectangle HAZ is equal to the ſquare of CB. Fig. 17.

For draw AF, BG parallel to the diameter CL: and ſince HF is to FC as HA to AZ, the rectangles HFC, HAZ are ſimilar; and HF is to HA as CG is to CB; therefore, ſince the rectangles HFC, HAZ are ſimilar, of which HF, HA are homologous ſides, and that the ſquares of CG, CB are ſimilar figures, the rectangle HFC [22. 6. Elem.] is to the rectangle HAZ as the ſquare of CG to the ſquare of CB: but the rectangle HFC is equal to the rectangle [1 cor. 17. 2.] EFD, that is, to the [19. 2.] ſquare of CG; and therefore the rectangle HAZ is alſo equal to the ſquare of CB.

Cor. If a ſtraight line HAZ touch an ellipſe, and meet two conjugate diameters CH, CZ; if the rectangle HAZ be equal to the ſquare of the ſemidiameter CB conjugated to CA, which paſſes through the point of contact; CH, CZ are two conjugate diameters.

PROP. XXII. THEOR.

If from a point of an ellipſe a ſtraight line be ordinately applied to a diameter, the rectangle contained by the ſegments of the diameter is to the ſquare of the ordinate as the diameter is to its *latus rectum.*

Fig. 18. Let F be a point in an ellipſe; from F draw FG, ordinately to the diameter AB; the rectangle AGB is to the ſquare of FG as the diameter AB to its *latus rectum.*

For let BH be equal to the *latus rectum*; and ſince the diameter AB, its conjugate DE, and *latus rectum* BH [9. def. 2.], are proportionals, AB is to BH [cor. 20. 6. Elem.] as the ſquare of AB to the ſquare of DE, that is, as the rectangle AGB to the ſquare of FG [15. 2.].

PROP. XXIII. THEOR.

If from a point of an ellipſe a ſtraight line be ordinately applied to a diameter, and from the vertex of the diameter a ſtraight line be drawn at right angles to the diameter, and equal to its *latus rectum*; the ſquare of the ordinate is equal to the rectangle applied to the *latus rectum*; which rectangle has for its breadth the abſciſſa between the ordinate and the vertex of the diameter, and is deficient by a figure ſimilar, and ſimilarly ſituated, to the figure contained by the diameter and the *latus rectum*.

Let F be a point in the ellipſe; from F draw FG, an ordinate to the diameter AB; from the vertex of AB draw a perpendicular BH, equal to the *latus rectum* of AB; the ſquare of FG is equal to the rectangle applied to BH; which rectangle has the abſciſſa BG for its breadth, and is deficient by a figure ſimilar, and ſimilarly ſi- Fig. 18.

tuated, to the rectangle BN, contained by the diameter and *latus rectum* BH.

Having joined AH, and from G drawn GK parallel to BH, and meeting AH in K, complete the parallelograms KLHM, ABHN; then, becauſe the rectangle AGB is to the ſquare of FG as [22. 2.] AB to BH, that is, as AG to GK, that is, as the [1. 6. Elem.] rectangle AGB to the rectangle KGB; the rectangle AGB is to the ſquare of FG as the ſame AGB to the rectangle KGB: and thus the ſquare of FG is equal to the rectangle KGB, applied to the *latus rectum* BH; which rectangle has for its breadth the abſciſſa GB, and is leſs than the rectangle BM, by the figure KLHM, ſimilar, and ſimilarly ſituated, to BN.

From the above-mentioned ſquare's being equal to the deficient rectangle, Apollonius named the curve line which is the ſubject of this ſecond book, the *Ellipſe*.

Cor. If from the vertex B of the diameter AB any ſtraight line BO be drawn equal to the *latus rectum*, though not at right angles to AB; join AO, and through G draw GP parallel to BO; the rectangle PGB is equal to the ſquare of FG.

PROP.

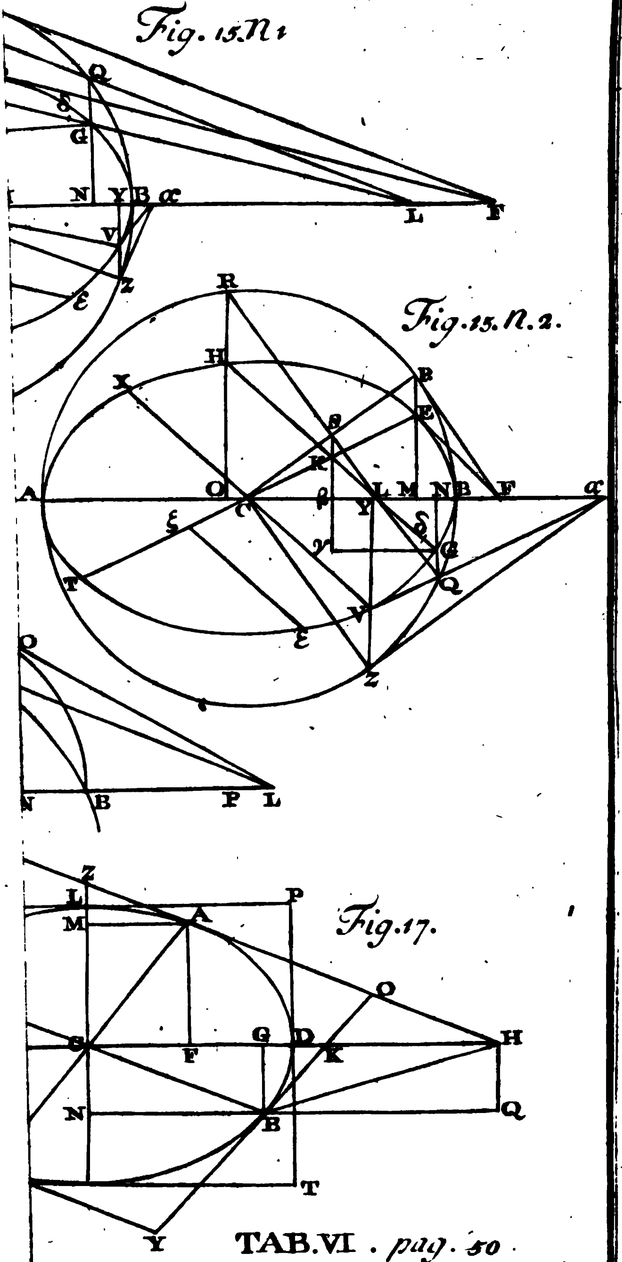

Fig. 15. N 1

Fig. 15. N. 2.

Fig. 17.

TAB. VI. *pag.* 50.

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PROP. XXIV. PROB.

Two unequal ſtraight lines which biſect each other at right angles, being given in poſition and magnitude, to deſcribe an ellipſe of which theſe may be the axes.

Let two unequal ſtraight lines AB, DE, which biſect each other at right angles in the point C, be given in poſition and magnitude; it is required to deſcribe an ellipſe of which theſe may be the axes. Fig. 19.

From the extremity D of DE, the leſs of the two, place DF equal to CB, the half of AB the greater; and from the centre D, with the diſtance DF, deſcribe a circle which will meet AB in two points; let theſe points be G and H; in which fix the ends of a ſtring of the ſame length with the ſtraight line AB, and deſcribe an ellipſe, as was directed in the firſt definition of this book; the ſtraight lines AB, DE will be the axes of this ellipſe.

For ſince the points G, H are the foci of the ellipſe, and that GC is [3. 3. Elem.]

equal to CH, the point C is its centre, [3. def. 2.]; and ſince CA or CB is equal to the length of the half of the ſtring, the ellipſe paſſes through the points A, B: it paſſes likewiſe through D [4. cor. 1. 2.], becauſe GD is equal to CA; and through E, becauſe CD is equal to CE.

PROP. XXV. PROB.

A ſtraight line being given in poſition and magnitude, and a point without it being given; to deſcribe an ellipſe, of which that ſtraight line ſhall be one of the axes, and which ſhall paſs through that given point; but the given point muſt be ſo ſituated, that a perpendicular drawn from it may fall between the extremities of the given ſtraight line.

Fig. 19. Let AB be the ſtraight line given in poſition and magnitude, and K the given point, ſo ſituated, that a perpendicular

drawn from it towards AB may fall between the extremities of AB; it is required to deſcribe an ellipſe which may have AB for one of the axes, and which may paſs through K.

Draw KL at right angles to AB, and find a ſtraight line DE ſuch *, that the ſquare of AB may be to the ſquare of DE as the rectangle ALB to the ſquare of KL; and place DE perpendicular to AB, and let them biſect each other, and with the ſame AB, DE, as the axes, deſcribe an ellipſe; this ellipſe will paſs through [2 cor. 15. 2.] the point K.

PROP. XXVI. PROB.

To find a diameter, the centre, the axes, and the foci of an ellipſe given in poſition.

Draw two ſtraight lines parallel to each other, and let them be terminated both

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* Find a mean proportional between AL, LB [13. and 17. 6. Elem.]; then to the three following ſtraight lines, viz. the mean proportional found, KL, and AB, find a fourth proportional [12. 6. Elem.]; this fourth proportional will be the DE wanted [22. 6. Elem.]

ways by the ellipſe; the ſtraight line which biſects them is [4. cor. 13. 2.] a diameter, and the point biſecting that diameter is [3. 2.] the centre.

Fig. 20. In order to find the axes, find the centre C, and in the ellipſe take any point A, and join CA; and from the centre C, and with the diſtance AC, deſcribe a circle AF: if this circle falls wholly without the ellipſe, CA is the greateſt of the ſemidiameters; and therefore the half of the [9. 2.] greater axis. Next, take any point D, and let a circle be deſcribed from the centre C, with the diſtance CD; if this circle falls wholly within the ellipſe, CD is the leaſt of the ſemidiameters; and therefore the half of the leſs axis [9. 2.]: or let the point G be taken; if the circle deſcribed from the centre C, with the diſtance CG, falls neither wholly without nor wholly within the ellipſe, the ſtraight line CG is the half neither of the greater nor of the leſs axis; the circle, conſequently, muſt meet the ellipſe again: let it meet it in H; and having joined GH, biſect it in K; CK will be one of the axes, and a ſtraight line drawn through the centre perpendicular to CK will be the other: for ſince GH is biſected by the diameter CK, it is parallel to the tangent AL drawn through the vertex of CK; and CKG is a right angle: therefore CAL is alſo a right angle; and therefore CKA is [4. cor. 10. 2.] one of the axes. The foci are found as in prop. 24.

PROP. XXVII. PROB.

Two conjugate diameters of an ellipſe being given in poſition and magnitude, to find the axes, and deſcribe the ellipſe.

Let AB, CD, be the given diameters; let them meet each other in the centre E. Suppoſe the problem ſolved, that is, let FG, HK be the axes to be found, and through A draw the ſtraight line AL parallel to CD; AL will touch [3. cor. 14. 2.] the ellipſe in A, and will be given [28. dat.] in poſition: let the ſame AL meet the axes in the points L, M; therefore the rectangle LAM is equal to the ſquare of CE, the half of the diameter conjugated to [21. 2.] AB: but CE is given, and, conſequently, its ſquare is given; there- Fig. 21.

fore the rectangle LAM is given: Let the rectangle EAN be equal to LAM; and becauſe EA is given in poſition and magnitude, AN is alſo given in poſition and magnitude; and becauſe the rectangles LAM, EAN are equal, the points L, E, M, N are [conv. 35. 3. Elem.] in the circumference of a circle: therefore, if EN is biſected in O, the centre of the circle will be in the ſtraight line OP, which is at right angles to EN [cor. 1. 3. Elem.]: but LEM being a right angle, the centre of the circle is likewiſe in [cor. 5. 4. Elem.] the ſtraight line LM: it is therefore in the point P, where OP, LM interſect each other: therefore the centre P is given, and the point E is given: hence the circle deſcribed from the centre P, with the diſtance PE, is given in [6. def. dat.] poſition; ſo likewiſe are the points L, M, where its circumference meets AB, a ſtraight line given in poſition: therefore the axes EL, EM are given in poſition: draw AQ at right angles to the axis FG; and becauſe AL touches the ellipſe, EQ, EF, EL are [17. 2.] proportionals; and EQ, EL are given: therefore EF is given in magnitude. It may in like manner be proved, that EK is given in magnitude; therefore the axes FG, HK are given in poſition and magnitude: and an ellipſe deſcribed through the point A, with the axis FG [25. 2.], will be an ellipſe, in which AB, CD are two conjugate diameters.

The compoſition is as follows. Produce EA to N, ſo that the rectangle EAN may be equal to the ſquare of CE: biſect EN in O, and draw OP at right angles to it; in the point P let OP meet the ſtraight line AL, which is parallel to CE; and from the centre P, diſtance PE, deſcribe a circle, and let AP meet its circumference in the points L, M: join EL, EM, and draw AQ at right angles to EL; and between EQ, EL find [13. 6. Elem.] a mean proportional EF, and make EG equal to EF: then, by the 25th propoſition of this book, deſcribe an ellipſe, of which FG may be one of the axes, and which may paſs through the point A: of this ellipſe AB, CD are conjugate diameters. For ſince AQ is perpendicular to the axis FG, and EQ, EF, EL proportionals, AL touches the ellipſe [18. 2.] in the point A; and becauſe CD is parallel to the tangent AL, it is in the ſame poſition with the diameter conjugated to AB; and the angle LEM being in a ſemicircle is a right angle; conſequently EM is the other axis, and ſo is conjugated to FG: hence the rectangle LAM is equal to the ſquare of the ſemidiameter conjugated to AE [21. 2.]: but the ſame rectangle LAM is equal [35. 3. Elem.] to the rectangle EAN, that is, by the conſtruction, to the ſquare of CE; therefore CE is the ſemidiameter conjugated to AE; the ellipſe therefore paſſes through C; and becauſe ED is equal to EC, and EB equal to EA, it paſſes likewiſe through the points D, B. Hence AB, CD are conjugate diameters in the ellipſe deſcribed.

PROP. XXVIII. PROB.

The poſition and magnitude of a diameter of an ellipſe being given, and the poſition being given of a ſtraight line which, from a given point in the ellipſe, is ordinately applied to that diameter; to deſcribe the ellipſe.

Let BA be the given diameter, to which RS, a ſtraight line given in poſition, is ordinately applied, from a given point R of the ellipſe. Fig. 21.

Biſect AB in the point E, and draw through E a ſtraight line parallel to RS, and in that parallel take equal ſtraight lines EC, ED, ſo that the rectangle ASB may be to the ſquare of RS as the ſquare of AE to the ſquare of EC or ED; then, by the preceding propoſition, deſcribe an ellipſe of which AB, CD may be conjugate diameters: this ellipſe will paſs through the [2. cor. 15. 2.] point R, and RS will be ordinately applied [4. cor. 14. 2.] to the diameter AB.

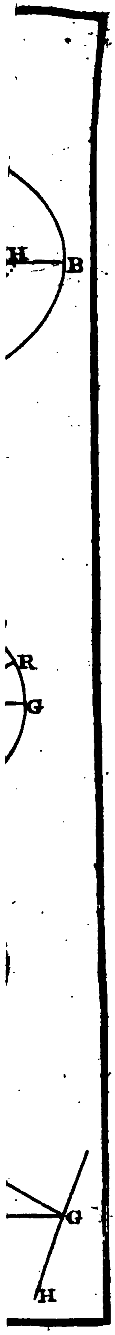
PROP. XXIX. THEOR.

If a cone cut by a plane paſſing through the axis, be cut alſo by another plane, meeting both the ſides, by which the angle at the vertex is contained, of the triangle through the axis, but neither parallel to the baſe of the cone, nor ſubcontrarily ſituated; if that other plane, and the plane in which the baſe of the cone is ſituated, meet in the direction of a ſtraight line perpendicular either to the baſe of the triangle through the axis, or to the ſtraight line which is in the ſame ſtraight line with that baſe; the line which is the common ſection of that other plane, and the conical ſurface, is an ellipſe, which has for one of its diameters the common ſection of the triangle through the axis, and of that ſame other plane.

Fig. 22. Let there be a cone, the vertex of which is the point A, and the baſe the circle BC; let it be cut by a plane through the axis, and let the ſection be the triangle ABC; let it be cut likewiſe by another plane, meeting both the ſides, by which the angle at A is contained, of the triangle

through the axis, but neither parallel to the baſe of the cone, nor ſubcontrarily ſituated; let the line DEF be the common ſection of this other plane with the conical ſurface; and let the common ſection of this plane, with the plane in which is the baſe of the cone, be GH perpendicular to BC: the line DEF is an ellipſe; and DF, the common ſection of the triangle through the axis, and this other plane, is one of its diameters.

In the ſection DEF take any point E, and through E to DF draw EK parallel to HG; next, through K draw LM parallel to BC: the plane, therefore, which paſſes through EK, LM is parallel to that through BC, GH, that is, to the baſe of [15. 11. Elem.] the cone: conſequently the plane through EK, LM [23. 1.] is a circle, of which LM is a diameter: but EK is perpendicular [10. 11. Elem.] to LM, becauſe GH is perpendicular to BG; the rectangle LKM is, therefore, equal [35. 3. Elem.] to the ſquare of EK. In like manner, any other point N being taken in the ſection DEF, if NO be drawn parallel to EK, or GH, and through O, PQ be drawn parallel to BC, it may be ſhewn, that the rectangle POQ is equal to the ſquare of NO: the ſquare of EK is, therefore, to that of NO, as the rectangle LKM to the rectangle POQ: but LK is to PO as DK is to DO; and KM is to OQ as KF is to OF; therefore the ratios compounded of theſe ratios are the ſame to one another; and therefore the rectangle LKM is to the rectangle POQ as the rectangle DKF is to the rectangle DOF [23. 6. Elem.]: and therefore the ſquare of EK is to that of NO as the rectangle DKF to the rectangle DOF. Deſcribe, therefore, an ellipſe [28. 2.] of which DF may be a diameter, and in which EK may be ordinately applied to DF: and becauſe the point E, by conſtruction, is in this ellipſe, the point N is likewiſe in it [3. cor. 15. 2.]. And the ſame thing may be demonſtrated with regard to all the points of the ſection DEF.



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BOOK III.

Of the HYPERBOLA.

DEFINITIONS.

I. IN a point E taken in a plane, the extremity E of a ruler EH is ſo fixed that the ruler is left free to revolve about the point as a centre: one end of a ſtring ſhorter than the ruler is fixed in the extremity H of the ruler, and the other end of it in the point F, which is in the ſame plane with the point E; but the diſtance between the points E, F muſt be greater than the exceſs of the length of the ruler above that of the ſtring: by means of a Fig. 1.

pin

pin G, as much of the ſtring as poſſible is applied to the ſide EH of the ruler; then with the ſtring ſo applied, and kept uniformly tenſe, the ruler is moved about the centre E: and thus by the point of the pin moving onwards with the ruler, a line is deſcribed, which is named the *hyperbola.*

But if the above order be reverſed, and the end E of the ruler be fixed in the point F, and the end F of the ſtring in the point E, and then a ſimilar operation be repeated, another line, oppoſite to the former, will be deſcribed, which in like manner is named the *hyperbola*; and both together are named *oppoſite hyperbolas.* Theſe lines may be extended beyond any given diſtance from the points E, F, if a ſtring be taken, the length of which exceeds that diſtance.

II. The points E, F are named the *foci.*

III. And the point C, which biſects the ſtraight line between the foci, is named the *centre of the hyperbola*, or of the *oppoſite hyperbolas.*

IV. Any ſtraight line paſſing through the centre, and meeting the hyperbolas, is named a *tranſverſe diameter*; and the points where a tranſverſe diameter meets the hyperbolas, are named its *vertices*. Alſo a ſtraight line which paſſes through the centre, and biſects any ſtraight line terminated by the oppoſite hyperbolas, but not drawn through the centre, is named a *right diameter*.

V. The diameter which paſſes through the foci is named the *tranſverſe axis*.

VI. If from either extremity A of the tranſverſe axis, a ſtraight line AD be placed equal to the diſtance between the centre C and either focus F, and from A as a centre, with the diſtance AD, a circle be deſcribed, meeting in the points B, *b* a ſtraight line drawn through the centre C at right angles to the tranſverſe axis; the ſtraight line B*b* is named the *ſecond axis*.

VII. Two diameters, each of which biſects all ſtraight lines parallel to the other, and terminated both ways by the hyperbola, or oppoſite hyperbolas, are named *conjugate diameters*.

VIII. When a ſtraight line not drawn through the centre, yet terminated both ways by the hyperbola, or oppoſite hyperbolas, is biſected by a diameter, it is ſaid to be *ordinately applied* to that diameter, or it is named, ſimply, an *ordinate* to the diameter.

meter. Alſo a diameter parallel to a ſtraight line ordinately applied to another diameter, is ſaid to be *ordinately applied* to this other diameter.

IX. A ſtraight line which meets the hyperbola in only one point, and which, being produced both ways, falls without the oppoſite hyperbolas, is ſaid to *touch* the hyperbola in that point.

PROP. I. THEOR.

If from a point in an hyperbola two ſtraight lines be drawn to the foci, the exceſs of the one above the other is equal to the tranſverſe axis.

Fig. 1. Let G be a point in an hyperbola, the exceſs of GE above GF is equal to the tranſverſe axis A*a*.

Let EGH repreſent the ruler, and FGH the ſtring, the pin by which the hyperbola is deſcribed being ſuppoſed to remain at G; from EH, FGH take away the common part GH; and the exceſs of GE above GF will be equal to the exceſs of the length of the ruler above that of the ſtring; and this conclusion will hold where-ever the point G ſhall be ſituated in the hyperbola. And ſince the points A, *a*, the vertices of the tranſverſe axis, are in the oppoſite hyperbolas, the exceſs of AE above AF, and the exceſs of *a*F above *a*E, are each of them equal to the exceſs of the length of the ruler above that of the ſtring, that is, to the exceſs of EG above GF; and therefore theſe two exceſſes are equal to each other: but let AF be added to each of the two ſtraight lines AE, AF; and the exceſs of AE above AF will be equal to the exceſs of FE above twice AF. In like manner, the exceſs of *a*F above *a*E will be equal to the exceſs of the ſame FE above twice *a*E; FE, therefore, exceeds twice AF by the ſame exceſs by which it exceeds twice *a*E: twice AF is, therefore, equal to twice *a*E; and therefore AF is equal to *a*E: therefore the exceſs of AE above AF is equal to the exceſs of AE above *a*E, that is, to the tranſverſe axis *a*A; and therefore the exceſs of EG above GF is equal to the ſame tranſverſe axis *a*A.

COR. And ſince AF is equal to *a*E and

CF to CE, CA is equal to C*a*, that is, the tranſverſe axis is biſected in the centre.

PROP. II. THEOR.

Let there be a point from which two ſtraight lines are drawn to the foci of oppoſite hyperbolas; if the exceſs of the one ſtraight line above the other be equal to the tranſverſe axis, that point is in one of the oppoſite hyperbolas.

Fig. 2. Let G be a point, whence GE, GF are drawn to the foci; if the exceſs of the one ſtraight line above the other be equal to the tranſverſe axis *a*A, the point G is in one of the oppoſite hyperbolas.

Of the two ſtraight lines let GF be the leſs; and from the centre F, diſtance FG, deſcribe a circle; let this circle meet FE in H; take GK equal to GF, and KE, by hypotheſis, will be equal to the tranſverſe axis A*a*: and becauſe FG, GE are together greater than FE; FG, GK are together greater than FA and *a*E together; therefore

fore (FG, or) FH, the half of FG, GK, is greater than FA, the half of FA, *a*E; and therefore the hyperbola, towards the point A, falls within the circle: and ſince it may be extended beyond any given diſtance from the focus F, it muſt cut the circle ſomewhere: Now it cuts the circle in the point G: For, if not, let it cut it in another point D, on the ſame ſide of the axis on which is the point G; and join DE, DF: the point D then is in the hyperbola; and therefore the exceſs of DE above DF is equal to the tranſverſe axis A*a*: but by hypotheſis, the exceſs of GE above GF is equal to the ſame A*a*; and FG is equal to FD: therefore EG is alſo equal to ED; which is contrary to prop. 7. b. 1. of Euclid; therefore the point G is in the hyperbola.

PROP. III. THEOR.

If two ſtraight lines be drawn from a point without an hyperbola to the foci, the exceſs of the one above the other is leſs than the tranſverſe axis; but if two ſtraight lines be drawn from a point within an hyperbola to the foci, the exceſs of the one above the other is greater than the tranſverſe axis. On the contrary, any point is without or within an hyperbola, according as the exceſs of two ſtraight lines drawn from that point is leſs or greater than the tranſverſe axis.

Fig. 2. From the point L without an hyperbola, let the two ſtraight lines LE, LF be drawn to the foci; the exceſs of the one above the other is leſs than the tranſverſe axis *Aa*: For ſince L is without and F within the hyperbola, the ſtraight line LF muſt meet the hyperbola. Let LF meet it in G, and join EG, and EL will be leſs than EG added to GL; the exceſs of EL above LF is, therefore, leſs than the exceſs of EG and GL together above the ſame LF, that is, than the exceſs of EG above GF, that is, than the tranſverſe axis *Aa*.

Next, from the point M within the hyperbola

perbola, draw ME, MF to the foci; ME will neceſſarily meet the hyperbola, becauſe the point M is within and the point E without it: let it meet it in N, and join NF; then, becauſe MF is leſs than MN added to NF, the exceſs of ME above MF is greater than the exceſs of the ſame ME above MN added to NF, that is, than the exceſs of NE above NF, that is, than the tranſverſe axis A*a*.

The laſt part of the propoſition is obvious.

COR. Hence, if through the vertex A of the tranſverſe axis, a ſtraight line be drawn at right angles to the tranſverſe axis, this ſtraight line is wholly without the hyperbola, and conſequently touches it.

For in the ſtraight line take any point Q, and join QF, QE; and in the axis place AR equal to AF, and join QR; then, becauſe AR is equal to AF, that is, to *a*E, RE is equal to the tranſverſe axis A*a*; alſo QF is equal to QR: but QE is leſs than QR added to RE; and therefore the exceſs of QE above QR or QF, is leſs than RE, that is, than the tranſverſe axis: Hence the point Q, and conſequently the ſtraight line AQ, is without the hyperbola.

PROP. IV. THEOR.

The ſquare of half the ſecond axis is equal to the rectangle contained by the ſtraight lines between either focus and the vertices of the tranſverſe axis.

Fig. 1. Let A*a* be the tranſverſe axis, C the centre, E, F the foci, and B*b* the ſecond axis, which, from the definition of it, is biſected in the centre; join AB; and becauſe AB, CF are [6. def. 3.] equal, the ſquares of AC, CB are together equal to the ſquare of CF, that is, to [6. 2. Elem.] the ſquare of AC added to the rectangle AF*a*: take away the common ſquare of AC, and there will remain the ſquare of CB equal to the rectangle AF*a*.

PROP. V. THEOR.

If from a point in an hyperbola a ſtraight line be drawn at right angles to the tranſverſe axis, and from that point a ſtraight line be drawn to the nearer (to that point) of the foci; half the tranſverſe axis is to the diſtance between that focus and the centre as the diſtance between the perpendicular and the centre is to the ſum of half the tranſverſe axis, and of the ſtraight line drawn from the point to that ſame focus.

Let G be the point in the hyperbola; Fig. 3. 4. from G draw GD perpendicular to the tranſverſe axis A*a*; and from the ſame point draw a ſtraight line GF to the nearer (to G) of the foci; CA, half the tranſverſe axis, is to CF, the diſtance between the centre and the focus F, as CD, the diſtance between the centre and the perpendicular, is to the ſum of half the tranſverſe axis and of GF.

Draw GF to the other focus, and in the axis *a*A produced place AH equal to GF, and from the centre G, diſtance GF, deſcribe a circle, meeting the axis *a*A again in K, and the ſtraight line EG in the points

L, M: and becauſe EF is the double of CF, and FK the double of FD, EK is the double of CD. Again, becauſe EL or A*a* is the double of CA, and LM the double of GF or AH, EM is the double of CH: but, on account of the circle, EL or A*a* is to EF as EK to EM [cor. 36. 3. and 16. 6. Elem.]; take the halves of theſe proportionals, and CA will be to CF as CD to CH.

PROP. VI. THEOR.

The ſame conſtruction remaining, if from the nearer (to the point in the hyperbola) of the vertices of the tranſverſe axis, and in this ſame axis produced, a ſtraight line be placed equal to the diſtance between that point and the nearer (to that nearer vertex) of the foci; the ſquare of the perpendicular is equal to the exceſs of the rectangle contained by the ſegments between the extremity (which is not the vertex of the tranſverſe axis) of that ſtraight line and the foci, above the rectangle contained by the ſegments between the perpendicular and the vertices of the tranſverſe axis.

Let A be the nearer (to the point G) of the vertices of the tranſverſe axis; from A place in A*a* produced AH equal to GF; the ſquare of the perpendicular GD is equal to the exceſs of the rectangle EHF, contained by the ſegments between the extremity H of the placed line AH and the foci, above the rectangle AD*a*, contained by the ſegments between the perpendicular GD and the vertices A, *a* of the tranſverſe axis. Fig. 3. 4.

For ſince the ſtraight line CH is cut into any two parts in the point A, the ſquares of CA, CH are together equal to twice the rectangle ACH, together with the ſquare of AH [7. 2. Elem.], that is (CA, CF, CD, CH being proportionals) to twice the rectangle FCD, together with the ſquare of AH or GF, that is, to twice the rectangle

FCD, together with the ſquares of FD, DG, that is, to the ſum of the ſquares of FC, CD and DG [7. 2. Elem.]: therefore the ſum of the ſquares of CA, CH is equal to the ſum of the ſquares of FC, CD, DG: but the ſum of the ſquares of CA, CH is equal [6. 2. Elem.] to the rectangle EHF, together with the ſum of the ſquares of CA, CF; and the ſum of the ſquares of FC, CD, DG is equal [6. 2. Elem.] to the rectangle AD*a*, together with the ſum of the ſquares of CA, DG, and FC: hence the rectangle EHF, together with the ſum of the ſquares of CA, CF, is equal to the rectangle AD*a* added to the ſum of the ſquares of CA, DG, FC: from theſe equals take away the common ſquares of CA, CF, and there will remain the rectangle EHF, equal to the rectangle AD*a*, together with the ſquare of DG.

PROP. VII. THEOR.

If from a point in an hyperbola a ſtraight line be drawn parallel to the ſecond axis, and meeting the tranſverſe axis; the ſquare of the tranſverſe axis is to the ſquare of the ſecond axis, as the rectangle contained by the ſegments (of the tranſverſe) intercepted between its vertices and the parallel is to the ſquare of the parallel.

Let G be the point in the hyperbola; from G draw GD parallel to the ſecond axis B*b*, and meeting the tranſverſe in D; the ſquare of A*a* is to that of B*b* as the rectangle AD*a*, contained by the ſegments between the vertices of the tranſverſe and the point D, is to the ſquare of GD. Fig 3. 4.

Having drawn two unequal ſtraight lines GE, GF to the foci, place, from the nearer (to the focus F) of the vertices of the tranſverſe axis, AH equal to GF the leſſer of them: then, becauſe CH, CD, CF, CA are proportionals, their ſquares are alſo proportionals; that is, as the whole, to wit, the ſquare of CH, is to the whole the ſquare of CD, ſo is the ſquare of CF to the ſquare of CA: therefore the remaining rectangle EHF is to the remaining rectangle AD*a* as the ſquare of CF to that of CA [cor. 19. 5. Elem.]:

Elem.]: and, by diviſion, the exceſs of the rectangle EHF above the rectangle AD*a*, is to AD*a* as the [6. 2. Elem.] rectangle AF*a* to the ſquare of CA; that is, by the preceding propoſition, and fourth of this book, the ſquare of GD is to the rectangle AD*a* as the ſquare of CB is to that of CA; and, inverſely, the ſquare of CA is to that of CB as the rectangle AD*a* is to the ſquare of GD.

COR. The ſquares of ſtraight lines drawn parallel to the ſecond axis from points in an hyperbola, or in oppoſite hyperbolas, are to one another, as the rectangles contained by the ſegments intercepted between thoſe ſtraight lines and the vertices of the tranſverſe axis, [ſee 1 cor. 6. 2.]

PROP. VIII. THEOR.

If from a point in an hyperbola a ſtraight line be drawn parallel to the tranſverſe axis, and meeting the ſecond axis; the ſquare of the ſecond axis is to that of the transverſe, as the ſum of the ſquares of half the ſecond axis, and of its ſegment between the parallel and the centre, is to the ſquare of the parallel.

From a point G of an hyperbola draw GN parallel to the tranſverſe axis A*a*, and meeting the ſecond axis B*b* in N; the inference is as already expreſſed in the propoſition. Fig. 4.

For, by the preceding, the ſquare of CA is to the ſquare of CB as the rectangle AD*a* is to the ſquare of GD: therefore, inverſely, and by propoſition 12. B. 5. Elem. the ſquare of CB is to the ſquare of CA, as the ſum of the ſquares of CB, GD is to the ſquare of CA, together with the rectangle AD*a*, that is, as the ſum of the ſquares of CB, CN is to the ſquare of CD or GN.

COR. Hence, if from two points of an hyperbola, or of oppoſite hyperbolas, there be drawn to the ſecond axis two ſtraight lines parallel to the tranſverſe axis, the ſquare of the one ſtraight line is to the

ſquare

ſquare of the other, as the ſum of the ſquares of half the ſecond axis, and of the diſtance between the one ſtraight line and the centre, is to the ſum of the ſquares of half the ſecond axis, and of the diſtance between the other ſtraight line and the centre.

PROP. IX. THEOR.

A ſtraight line terminated both ways by an hyperbola, or oppoſite hyperbolas, and parallel to either axis, is biſected by the other axis, or, what is the ſame thing, the axes are conjugate diameters.

Fig. 5. Firſt, let the ſtraight line DE be parallel to the ſecond axis B*b*, and meet the tranſverſe in F; and thus the ſquare of DF is to the ſquare of EF as the rectangle AF*a* is to the [cor. 7. 3.] rectangle AF*a*; hence DF, FE are equal.

Next, let DG be parallel to the tranſverſe axis A*a*, and meet the ſecond axis B*b* in K; and thus the ſquare of DK is to the ſquare of KG, as the ſum of the ſquares of CB, CK is to the ſum of the ſame ſquares [cor. preced.] of CB, CK: hence DK, GK are equal.

PROP. X. THEOR.

A ſtraight line terminated both ways by an hyperbola, or oppoſite hyperbolas, and biſected by either axis, is parallel to the other axis.

Firſt, let DE be biſected by the tranſverſe axis in F, and let DK, EL be drawn parallel to the ſame axis, and meeting the ſecond axis in the points K, L; then, becauſe DF, FE are equal, KC, CL are alſo equal: but the ſquare of DK is to that of EL as the ſquare of CB, together with the ſquare of CK, is to the ſquare of CB, added to the ſquare of CL; DK, EL are, therefore, equal, and they are parallel: hence DE, KL are alſo parallel. [31. 1. Elem.]. Fig. 5.

Next, let DG be biſected by the ſecond axis in the point K, and let DF, GM be

drawn

drawn parallel to the ſame ſecond axis, and meeting the tranſverſe axis in F, M; then, becauſe DK, GK are equal, FC, CM are equal; and, of conſequence, FA, *a*M are equal: but the ſquare of DF is to the ſquare of GM as the rectangle AF*a* to the rectangle AM*a*; and the rectangles AF*a*, AM*a* are equal; therefore the ſtraight lines DF, GM are equal, and they are parallel; therefore DG, FM are parallel [31. 1. Elem.].

Cor. It is manifeſt from the demonſtration, that the ſtraight lines DF, GM, which are parallel to either axis B*b*, and cut off, between the centre and the points where they meet the other axis, equal ſegments FC, MC, are equal. In the ſame manner, DK, EL are equal; which are parallel to the axis A*a*, and cut off the equal ſegments CK, CL.

If, on the other hand, DF, GM are equal to each other, and parallel to B*b*, they cut off equal ſegments FC, MC. In like manner, if DK, EL be equal to each other and parallel to A*a*, they cut off equal ſegments CK, CL.

PROP

PROP. XI. THEOR.

Any ſtraight line perpendicular to the tranſverſe axis, and meeting it below the vertex, meets the hyperbola in two points.

Perpendicular to the tranſverſe axis A*a*, and meeting it in C, below the vertex A; DC meets the hyperbola in two points: Let E, F be the foci; and from C, place CG equal to CF, the diſtance between C and the nearer (to C) of the foci; and from the other focus EK, equal to the tranſverſe axis A*a*. If, therefore, the point C be below the focus F, it is evident, that EK is leſs than EG: but in the caſe where the point C is above F, ſince A*a*, EK are equal, AK is equal to *a*E, that is, to AF: and, by hypotheſis, FC is leſs than FA; twice FC is, therefore, leſs than twice FA, that is, FG is leſs than FK; and thus EK is leſs than EG: make then EK to EF as EG is to a fourth proportional EH; and ſince EK is leſs than each of the two EF, EG, and, of conſequence, much leſs than EH, EK, EH are [25. 5. Elem.] together greater than Fig. 6. 7. Fig. 6. Fig. 7. Fig. 6. 7.

than EF, EG together. From theſe unequals take away twice EK, and KH will be greater than KF and KG together, that is, than twice KC; for CF is equal to CG: hence, if KH is biſected in L, KL will be greater than KC; and therefore the point L falls below the ſtraight line CD; and a circle deſcribed from the centre E, with the diſtance EL, will neceſſarily meet CD in two points D, *d*. Deſcribe, from the centre D, diſtance DF, another circle, which will paſs through the point G; join DE, and let this circle meet it in the points M, N; and ſince EK is to EF as EG to EH, the rectangle HEK is equal to the rectangle FEG, that is, to the rectangle MEN [36. 3. Elem.]; and ED, EL are equal: and thus their ſquares are equal: from which take away the equal rectangles MEN, HEK, and the remaining ſquare of DM or DF is equal to the remaining ſquare of KL [6. 2. Elem.]: hence DM and KL are equal; which being taken from the equals ED, EL, the remainder EM is equal to the remainder EK, or the tranſverſe axis A*a*; and EM is the exceſs of DE above DF: therefore the point D is in the

hyperbola

rbola. In like manner it may be de- ſtrated, that the point *d* is in the hy- ola.

DEFINITION X.

f through one of the vertices of the aſverſe axis a ſtraight line be drawn e- al and parallel to the ſecond axis, and ected by the tranſverſe axis; the ſtraight es drawn through the centre and the tremities of the parallel are named the *ymptotes.* Fig. 8.

Cor. 1. The aſymptotes of two oppoſite yperbolas are common to both.

For let CD, CE be the aſymptotes of the yperbola AF, and draw through the ver- ex A of the tranſverſe axis the ſtraight line DAE parallel to the ſecond axis B*b*, and through the other vertex *a* the ſtraight line *dae* parallel to DE; then becauſe CD, CE are aſymptotes, DA, AE are each of them equal and parallel to CB, the half of the ſecond axis: and becauſe DE, *de* are parallel, and CA, C*a* equal, *ad*, *ae* are equal and parallel to AD, AE; conſequently they are equal and parallel to half the ſecond axis: therefore C*d*, C*e* are aſymptotes of the oppoſite hyperbola *a*.

Cor. 2. The aſymptotes are parallel to ſtraight lines joining the extremities of the axes; for if AB, *b*A be joined, CE, CD are parallel to them [33. 1. Elem.]

PROP. XII. Theor.

The aſymptotes do not meet the hyperbola.

Fig. 8. Let there be an hyperbola, the tranſverſe axis of which is A*a*, and the centre C, and through A draw a ſtraight line perpendicular to CA; and in this perpendicular take AD, AE, equal, each of them, to hal the ſecond axis; join CD, CE; which are therefore the aſymptotes; and, if poſſible, let CD meet the hyperbola in F, and through F draw a ſtraight line parallel t DA, and meeting the axis A*a* in G; an ſince the rectangle AG*a* is to the ſquare o GF, as the ſquare of CA is [7. 3.] to tha of CB or AD, that is, as the ſquare of C is to that of GF, the rectangle AG*a* is equal to the ſquare of CG; which is ab

ſurd: the aſymptote, therefore, meets not the hyperbola in F; nor, as it may be ſhewn [6. 2. Elem.], does it meet the hyperbola in any other point.

PROP. XIII. THEOR.

If through a point of an hyperbola a ſtraight line be drawn parallel to the ſecond axis, and meeting the aſymptotes; the rectangle contained by its ſegments intercepted between the aſymptotes and that point, is equal to the ſquare of half the ſecond axis.

Let F be a point in the hyperbola; through F draw KFL parallel to the ſecond axis, and meeting the aſymptotes in the points K, L; the rectangle, &c. Fig. 8.

Through the vertex A of the tranſverſe axis draw DAE, meeting the aſymptotes in the points D, E; and let KL meet the ſame axis in G: therefore AD, AE are each of them equal and parallel to half the ſecond axis. To the ſecond axis draw the ſtraight line FM parallel to CA; and, by the 8th propoſition of this book, the ſquare of CB or AD will be to the ſquare of CA as the ſum of the ſquares of CB, CM is to the ſquare of FM or GC; and the ſquare of AD is to the ſquare of AC as the ſquare of KG to the ſquare of GC; therefore the ſum of the ſquares of CB, CM is to the ſquare of GC as the ſquare of KG is to the ſame ſquare of GC: the ſum of the ſquares of CB, CM is, therefore, equal to [9. 5. Elem.] the ſquare of KG: from theſe equals take the equal ſquares of CM, FG, and the remaining ſquare of CB is [5. 2. Elem.] equal to the remaining rectangle KFL. In like manner, if KL meets the hyperbola again in H, it may be ſhewn, that the rectangle KHL is equal to the ſquare of CB.

Cor. Hence if in a ſtraight line KL terminated by the aſymptotes, and parallel to the ſecond axis, there be taken a point F, ſo ſituated, that the rectangle KFL may be equal to the ſquare of the ſecond axis; that point is in the hyperbola.

PROP.

PROP. XIV. Theor.

If a ſtraight line meeting an hyperbola in two points, meets alſo the aſymptotes; the rectangle contained by the ſegments between the aſymptotes and the one point is equal to that contained by the ſegments between the ſame aſymptotes and the other point: Alſo if a ſtraight line meeting two oppoſite hyperbolas in two points, meets likewiſe the aſymptotes; the rectangle contained by the ſegments between the aſymptotes and the one point, is equal to that contained by the ſegments between the ſame aſymptotes and the other point: And in both caſes the ſtraight lines between the aſymptotes and the points are equal.

Fig. 9. Let AB be a ſtraight line meeting the hyperbola, or oppoſite hyperbolas, in the points A, B, and the aſymptotes in C, D; the rectangles CAD, CBD are equal; and CA, BD are equal.

Through the points A, B draw ſtraight lines parallel to the ſecond axis, and meeting the aſymptotes in E, F and in G, H: and ſince, by the preceding propoſition, the rectangles EAF, GBH are each of them equal to the ſquare of half the ſecond axis, they are equal to each other; therefore, as EA to GB ſo is BH to AF: but the triangles being equiangular, as EA to GB ſo is CA to CB; and as BH to AF ſo is BD to AD: therefore as CA to CB ſo is BD to AD: the rectangle, therefore, CAD is equal to the rectangle CBD: Take away * or add † the common rectangle AC, BD, and the rectangle CAB is equal to the rectangle ABD; and therefore the ſtraight lines AC, BD are equal [1. 6. Elem.].

Cor. If from two points A, L in an hyperbola,

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* In the caſe where the points are in the ſame hyperbola.

† In the caſe where the points are in oppoſite hyperbolas.

perbola, to either aſymptote KC ſtraight lines AM, LN be drawn parallel to the other aſymptote; there being drawn to any point B in the hyperbola ſtraight lines AB, LB, which meet the aſymptote KC in C, O; CO, MN are equal: for let AB, LB meet the aſymptote KP in the points D, P, and to the other aſymptote KC draw BQ parallel to the aſymptote KP: ſince AC, BD are equal, as alſo OL, BP; CM, QK are equal, as alſo ON, QK; CM, therefore, is equal to ON; and MO being common, CO, MN are equal.

PROP. XV. Theor.

If through two points in an hyperbola, or in oppoſite hyperbolas, two parallel ſtraight lines be drawn which meet the aſymptotes; the rectangles contained by their ſegments between the points and the aſymptotes are equal.

Let A, B be two points in an hyperbola, or in oppoſite hyperbolas; through theſe points Fig. 10. n. 1.

points draw CD, EF parallel to each other, and meeting the aſymptotes OC, OD in the points C, D and E, F; the rectangles GAD, EBF are equal.

Through the points A, B to the aſymptotes draw the ſtraight lines GAH, KBL parallel to the ſecond axis; and becauſe the rectangles GAH, KBL are each of them equal [13. 3.] to the ſquare of half the ſecond axis, they are equal to each other; therefore GA is to KB as BL to AH: but the triangles GAC, KBE being equiangular, GA is to KB as CA to EB; and the triangles LBF, HAD being equiangular, BL is to AH as BF to AD; hence CA is to EB as BF to AD; and hence the rectangles CAD, EBF are equal.

Fig. 10. n. 2. Cor. 1. And if through the centre a ſtraight line AOM be drawn meeting both the hyperbolas, and parallel to the ſtraight line BEF, the ſquare of either ſegment AO, intercepted between the centre and either hyperbola, is equal to the rectangle EBF. The demonſtration is the ſame as in the propoſition.

Cor. 2. Hence any ſtraight line drawn

through the centre, and terminated by oppoſite hyperbolas, is biſected in the centre.

Fig. 10. n. 1.

COR. 3. If CD, EF meet an hyperbola, or its oppoſite hyperbola again in the points M, N, the rectangle ACM, or ADM, is equal to the rectangle BEN or BFN; for AC, MD are equal, as alſo BF, NE.

Fig. 10. n. 2.

COR. 4. And ſince it has been proved, that the ſquare of the ſemidiameter AO or OM is equal to the rectangle EBF, that is, to BEN; BE, for this reaſon, is to AO as AO to EN; and conſequently BN is greater than [25. 5. Elem.] twice AO, that is, than AM; that is, a tranſverſe diameter is leſs than any ſtraight line parallel to it, and terminated in oppoſite hyperbolas.

COR. 5. If in a ſtraight line BN terminated by the hyperbolas, there be taken points E, F, which make each of the rectangles BEN, BFN equal to the ſquare of the ſemidiameter AO, which is parallel to BN; the points E, F are in the aſymptotes.

PROP.

PROP. XVI. THEOR.

If from a point in an hyperbola to the aſymptotes any two ſtraight lines be drawn, to which other two ſtraight lines drawn to the aſymptotes from any other point in the ſame or oppoſite hyperbola, are parallel; the rectangle contained by the former ſtraight lines is equal to that contained by the latter.

Fig. 11. Let A, B be points in an hyperbola, or in oppoſite hyperbolas; through A draw the ſtraight lines AC, AD to the aſymptotes, and through B draw BE, BF parallel to AC, AD; the rectangle CAD is equal to the rectangle EBF.

Draw through the points A, B to the aſymptotes the ſtraight lines GAH, KBL parallel to the ſecond axis, and the propoſition may be demonſtrated in the ſame words with the preceding.

COR. 1. Hence, if from two points in an hyperbola, or oppoſite hyperbolas, to one or both of the aſymptotes, two ſtraight lines be drawn parallel to the other aſymptote, or to both of them; the rectangle contained by the one parallel and the ſegment between it and the centre, is equal to that contained by the other parallel and the ſegment between it and the centre.

Let A, B be the points; through them draw AC and BE, or BF, parallel to the aſymptotes; the rectangle contained by the parallel AC, and the ſegment CO between AC and the centre, is equal to the rectangle BEO or BFO: for if the parallelograms ACOD, BEOF be completed, the rectangles CAD, EBF, that is, the rectangles ACO, BEO, will be equal.

COR. 2. And ſince the rectangles CAD, EBF are equal, AC is to BE as BF to AD; and the parallelograms ACOD, BEOF being equiangular, are therefore equal. [14. 6. Elem.].

PROP. XVII. THEOR.

Any ſtraight line drawn through the centre, and which paſſes within the angle contained by the aſymptotes, meets the hyperbola.

Fig. 12. Let AB, AC be the aſymptotes, and AD the half of the tranſverſe axis, and let AE be any ſtraight line drawn through the centre and paſſing within the angle BAC; AE meets the hyperbola. For if AE meets not the hyperbola, through D draw BDC parallel to the ſecond axis, and meeting the aſymptotes in B, C; draw alſo DF parallel to AB, and let DF meet AE in F; and having taken BG equal to DF, join GF, which will be equal and parallel to BD, and will therefore meet the tranſverſe axis at right angles, and ſo will cut the hyperbola [11. 3.]: let it cut it in H on the ſame ſide of AD on which is the point F; and let it meet the other aſymptote in K: ſince, therefore, the point F is without the hyperbola, GH is greater than GF, that is, than BD; and HK is greater than DC:

therefore the rectangle GHK is greater than the rectangle BDC, that is, than the ſquare of BD: but, by the 13th of this book, the rectangle GHK is equal to the ſquare of BD; which is abſurd: therefore AE neceſſarily meets the hyperbola.

Cor. If from the centre a ſtraight line OA be drawn within the angle contained by the aſymptotes, and the ſquare of that ſtraight line be equal to the rectangle EBF, contained by the ſegments of any ſtraight line parallel to OA, which are intercepted between the point B, where that parallel meets the hyperbola, and the points E, F where it meets the aſymptotes; the point A is in one of the hyperbolas: for, according to the propoſition, the ſtraight line OA neceſſarily meets the hyperbola: if, therefore, it meets not the hyperbola in A, let it meet it, if poſſible, in P; and then the ſquare of OP will be equal to the [1 cor. 15. 3.] rectangle EBF, that is, to the ſquare of OA; which is abſurd: therefore the point A is in the hyperbola. Fig. 10. n. 2.

PROP. XVIII. [*Prop.* 13. *B.* 2. *Apoll.*]

If within the angle contained by the aſymptotes any ſtraight line be drawn parallel to either of the aſymptotes, it meets the hyperbola in one point only, and paſſes within the hyperbola.

Fig. 12. Let there be an hyperbola, the aſymptotes of which are AL, AM; take any point N, and through N draw NO parallel to AL; NO will meet the hyperbola. For, if poſſible, let NO not meet the hyperbola; and in the hyperbola take any point P, through which draw PQ, PM parallel to AM, AL, and make the rectangle ANO equal to MPQ; and having joined AO, produce it; AO will meet the [17. 3.] hyperbola: let it meet it in the point R, and through R draw RS, RT parallel to AM, AL; therefore the rectangle MPQ is equal to the [16. 3.] rectangle TRS: but the rectangle ANO is made equal to the ſame MPQ; the rectangle, therefore, TRS, that is, the rectangle ATR, is equal to ANO;

which is impoſſible, becauſe RT is greater than NO, and AF greater than AN: therefore NO muſt meet the hyperbola. Let it meet it in the point V, and it is to be proved that it does not meet it in any other point: For, if poſſible, let it meet it likewiſe in X, and through V, X draw VY, XL parallel to AM; therefore the rectangle NVY is equal to the rectangle NXL; which is abſurd: therefore NO meets the hyperbola no where but in the point V. Laſtly, in the ſtraight line NV produced, take the point X, and through X draw a ſtraight line parallel to AN, and let this parallel meet AY in L and the hyperbola in Z; therefore the rectangle XLA is greater than VYA, that is, than ZLA; therefore LX is greater than LZ; and thus the point X is within the hyperbola.

COR. 1. It appears from the demonſtration, that a ſtraight line drawn through the centre, and paſſing between the aſymptotes, meets the hyperbola in one point only: for ſhould AR meet the hyperbola in another point O, the rectangles RTA, ONA would be equal; which is abſurd.

COR. 2. And if a ſtraight line meet an

hyperbola, or oppoſite hyperbolas, in two points, it meets both the aſymptotes; for if it were parallel to the one of the aſymptotes, it would meet the hyperbola in only one point.

COR. 3. And if a ſtraight line touch an hyperbola it meets both the aſymptotes; for if it were parallel to the one of them, it would paſs within the hyperbola; which is abſurd.

COR 4. If through the point N in one aſymptote a ſtraight line NO be drawn parallel to the other, and in this ſtraight line, and within the angle within which the hyperbola is deſcribed, a point V be taken, making the rectangle VNA, contained by a ſtraight line between the aſymptote AM and the point V, and by the ſegment between that ſtraight line and the centre, equal to the rectangle PMA, contained by a ſtraight line drawn from any point P of the hyperbola, ſo as to be parallel to the aſymptote AL, and by the ſegment between this parallel and the centre; the point V is in the hyperbola: for if NO meet not the hyperbola in V, let it, if poſſible, meet it in X; the rectangle XNA is, therefore, equal to the rectangle PMA, that is, to the rectangle VNA; which is abſurd: therefore the point V is in the hyperbola.

PROP. XIX. THEOR.

Fig. 13.

If through a point A of an hyperbola a ſtraight line be drawn meeting both the aſymptotes in the points B, C; if from either of the points C another ſtraight line CD be placed equal to the ſtraight line intercepted between the point A in the hyperbola and the remaining point B, ſo that the extremity D of the ſtraight line CD, and the point A in the hyperbola, may be either both between or both beyond the points B, C; the point D, in the one caſe, is in the hyperbola in which the point A is; but in the other, it is in the oppoſite hyperbola.

Let G be the centre of the hyperbolas, and through A, D, to either aſymptote GB, draw ſtraight lines AE, DF parallel to the remaining aſymptote; and, becauſe of the parallels, BA is to DC as BE to FG: but BA, DC are equal; therefore BE, FG are equal; and ſo BF, EG are alſo equal: and becauſe of the equiangular triangles AE is to DF as BE to BF, that is, as FG to EG the rectangle AEG is therefore equal to the rectangle DFG, and the point A is in the hyperbola; and therefore the point D (GF being one of the aſymptotes) is alſo in the hyperbola [4 cor. preceding].

PROP. XX. THEOR.

If a ſtraight line cuts both the ſtraight lines containing the angle adjacent to that within which an hyperbola is deſcribed, it meets each of the hyperbolas in only one point.

Fig. 13. Let there be an hyperbola, the aſymptotes of which are GB, GC, and let the ſtraight line BC cut them in the points

B,

B, C; and having taken in the hyperbolas any point H, through that point draw HIK parallel to BC, and meeting the aſymptotes in I, K; and to the ſtraight line BC apply a rectangle equal to the rectangle IHK, and let this applied rectangle exceed by a ſquare [29. 6. Elem.]; and let A, D be the points of application; A, D are points in the hyperbolas. For through A, H draw the ſtraight lines AE, AN, and HL, HM parallel to the aſymptotes; and becauſe the rectangles BAC, KHI are equal, BA is to KH as HI to AC: but, on account of the equiangular triangles, BA is to KH as AE to HL; and HI is to AC as HM to AN: therefore AE is to HL as HM to AN; and therefore the rectangle EAN, or AEG, is equal to the rectangle MHL, or HLG; and the point H is in the hyperbola; therefore the point A is alſo in the ſame or in the oppoſite hyperbola [4. cor. 18. 3.]; and in the ſame manner D is in the hyperbola oppoſite to the hyperbola in which the point A is: And it is manifeſt, that BC does not meet the hyperbolas in any other point.

COR. Hence if a ſtraight line BC cut

both the ſtraight lines containing the angle adjacent to that within which the hyperbola is deſcribed, and in BC produced a point A be taken, which makes the rectangle BAC equal either to KHI, contained by the ſegments of any ſtraight line HK parallel to BC, which ſegments are intercepted between the point H where HK meets the hyperbola, and the points K, I where it meets the aſymptotes; or to the ſquare of the ſemidiameter parallel to BC; the point A is in one of the hyperbolas.

PROP. XXI. THEOR.

Let a ſtraight line cut both the ſtraight lines within which an hyperbola is deſcribed; if the ſquare of the half of it be not leſs than the rectangle contained by the ſegments of another ſtraight line drawn parallel to it through any point of the hyperbola, which ſegments are between the hyperbola and the aſymptotes; that ſtraight line meets the hyperbola.

Let there be an hyperbola, the aſymptotes of which are OC, OD, and let a ſtraight line GH cut them; GH meets the hyperbola, if the ſquare of its half be not leſs than the rectangle mentioned in the propoſition. Fig. 11

Having taken in the hyperbola any point M, through that point draw a ſtraight line parallel to GH, and meeting the aſymptotes in K, L; and to the ſame GH apply a rectangle equal to the rectangle KML, and deficient by a ſquare; which, from the determination, is poſſible [27. 28. 6. Elem.], and let A be the point of application; this point will be in the hyperbola: for if the ſtraight lines AC, MN be drawn through the points A, M parallel to the aſymptotes, the rectangles ACO, MNO will be equal; becauſe the rectangle GAH is equal to the rectangle KML; and the point M is in the hyperbola: therefore the point A is alſo in it. In like manner it may be proved, that the other point of application is in the hyperbola: but if the ſquare of the half of

GH be equal to the rectangle KML, the point biſecting GH is the only point of GH that is in the hyperbola.

COR. Hence, if a ſtraight line GH cut the aſymptotes OK, OL of an hyperbola, and in GH a point A be taken, making the rectangle GAH equal either to the rectangle KML, contained by the ſegments of any ſtraight line KL parallel to GH, which ſegments are intercepted between the point M where KL meets the hyperbola, and the points K, L, where it meets the aſymptotes; or to the ſquare of the ſegment of the tangent parallel to GH, which ſegment is between the aſymptote and point of contact; the point A is in one of the hyperbolas.

PROP. XXII. [*Prop.* 14. *B.* 2. *Apoll.*]

An aſymptote and the hyperbola, produced without end, continually * approach; and the diſtance between them becomes leſs than any given diſtance.

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* See propoſition 12.

Let there be an hyperbola, the aſymptotes of which are AB, AC, and let D be the given diſtance; and in the hyperbola let E, F be two points, through which draw GEH, CFL parallel to each other, and meeting the aſymptotes in the points G, H and C, L; join AE, and let it meet CL in K: then, becauſe the rectangle GEH [15. 3.] is equal to the rectangle CFL, LF is to HE as EG is to FC: but LF is greater than HE, becauſe KL is greater than HE; therefore EG is alſo greater than FC. In like manner it may be proved, that the ſtraight lines which follow are ſucceſſively leſs than FC. Take then a diſtance GM leſs than the diſtance D, and through M draw MN parallel to AC; therefore MN will meet [18. 3.] the hyperbola: Let it meet it in N, and through N draw ONB parallel to GH; ON is, therefore, equal to GM, and therefore leſs than the diſtance D. Fig. 14.

PROP.

PROP. XXIII.

If a ſtraight line intercepted between the aſymptotes meets the hyperbola, and is biſected in the point where it meets it, it touches the hyperbola; and if it touches the hyperbola, it is biſected in the point of contact.

Fig. 15. n. 1. Let there be an hyperbola the aſymptotes of which are AB, AC, and let a ſtraight line BC, terminated by the aſymptotes, meet it in the point D, and be biſected in D; the ſtraight line BC touches the hyperbola.

Through D draw DE parallel to the one aſymptote AC, and meeting the other in E; and in BC take any point G, through which draw GH parallel to DE; GH will meet the hyperbola in [18. 3.] ſome point F: then, becauſe BD, DC are equal, BE, EA are alſo equal; and, becauſe of the equiangular triangles, BE is to ED as BH to HG; therefore the rectangle BEA is to the [1. 6. Elem.] rectangle DEA as the rectangle

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angle BHA to GHA: but the rectangle BEA is [5. 2. Elem.] greater than BHA; therefore the rectangle DEA is greater [14. 5. Elem.] than the rectangle GHA; that is, (F being a point in the hyperbola), the rectangle FHA is greater than the rectangle GHA; and therefore FH is greater than HG; therefore the point G is without the hyperbola; and therefore the ſtraight line BC touches the hyperbola in the point D.

OTHERWISE.

If a ſtraight line LM, terminated by the aſymptotes, is biſected by the hyperbola in the point D, it touches the hyperbola. Fig. 15. n. 1.

It is plain that LD paſſes not within the hyperbola; for if it paſſed within the hyperbola, it would neceſſarily meet the hyperbola again in another point, becauſe the points L, M are without the hyperbola: but it is impoſſible for it to meet the hyperbola in any other point but D; for, if poſſible, let it meet it in N; therefore NM is [14. 3.] equal to DL, that is, according to the hypotheſis, to DM; which is abſurd: therefore LM falls not within the hyperbola, nor meets it any where but in the point D; and therefore LM touches it in D. On the contrary, if the ſtraight line LM, terminated by the aſymptotes, touch the hyperbola in D, it is biſected in the point of contact.

For if LD, DM are unequal, from DM the greater take away MN equal to LD the leſs; therefore the point N is [19. 3.] in the hyperbola; and therefore, contrary to the hypotheſis, LM cuts the hyperbola.

Fig. 15. n. 1. Cor. 1. Hence through the ſame point D of an hyperbola, only one ſtraight line can be drawn touching the hyperbola.

Let D be a point in the hyperbola, and through that point to the aſymptote AB draw a ſtraight line DE parallel to the other; and take EB equal to EA, and having joined BD, let it meet the aſymptote AC in C: then, ſince BE, EA are equal, BD, DC are equal; BC, therefore, touches the hyperbola in D. And no other ſtraight line touches it in D: for, if poſſible, let LDM alſo touch it; then, ſince BE, EA are equal, LE, EA are unequal; and conſequently LD, DM are unequal: therefore LM does not touch the hyperbola.

COR. 2. Hence a method is pointed out, by which, if the aſymptotes AB, AC of an hyperbola be given in poſition, a ſtraight line BC can be drawn which may touch the hyperbola in a given point D.

COR. 3. If through the vertices of a tranſverſe diameter two ſtraight lines be drawn touching the hyperbolas, they are parallel to each other. Let AC, BC be the aſymptotes, and let AOB, QPR touch the hyperbolas in the vertices of the tranſverſe diameter OCP; AB, QR are parallel. There being drawn to either aſymptote AC the ſtraight lines OS, PT parallel to the other, the triangles SCO, TCP are equiangular: and ſince, by the propoſition, AO, OB are equal, AS, SC are alſo equal; and, in like manner, QT, TC are equal: and CO is to OP as CS to CT; and, conſequently, as CA to CQ; therefore the triangles OCA, PCQ are equiangular; and therefore OA, PQ are parallel. Fig. 15. n. 2.

COR. 4. And if a ſtraight line be drawn parallel to a tangent, and meeting the hyperbola, the ſquare of the ſegment (of the tangent) between the point of contact and either of the aſymptotes, is equal to the rectangle contained by the ſegments of the parallel which are between the aſymptotes and the point where it meets the hyperbola; for this rectangle is equal to the rectangle contained by the ſegments of the tangent [15, 3.] which are between the point of contact and the aſymptotes, that is, to the ſquare of the ſegment (of the tangent) between the point of contact and either of the aſymptotes.

PROP. XXIV. PROB.

Fig. 15. p. 1. The aſymptotes AB, AC of an hyperbola, and a point F in the ſame, being given in poſition, to draw a ſtraight line which ſhall touch the hyperbola, and be parallel to a ſtraight line KO given in poſition, and which cuts both the aſymptotes of the hyperbola, or oppoſite hyperbolas.

Suppoſe the problem ſolved, and let BC

be parallel to KO, and touch the hyperbola in D; and having joined AD, let AD meet KO in P; draw FRQ parallel to AD, and meeting the aſymptotes in Q, R; and ſince the ſtraight line BC touches the hyperbola in D, BD is equal to BC [23. 3.]; and conſequently KP is equal to PO; and KO is given in poſition and magnitude; therefore KP is given, and the point P; and the point A is given; therefore the ſtraight line PAD is given in poſition: the ſquare of AD is equal to [1. cor. 15. 3.] the rectangle QFR: and ſince FRQ is given in poſition [28. dat.], being drawn through a given point F, parallel to a ſtraight line AD given in poſition, and that AB, AC are given in poſition; therefore FQ, FR are [25. 26. dat.] given; therefore the rectangle QFR is given; conſequently the ſquare of AD is given; and therefore AD is given in magnitude: but the point A is given in poſition; therefore the point D is alſo given [27. dat.]; and therefore the ſtraight line BDC is given in poſition.

The compoſition is this: Let KO be biſected in P; and having joined AP, draw through the point F a ſtraight line FRQ parallel to AP, and meeting the aſymptotes in the points R, Q: again, in AP produced, and in either direction from the centre, let there be placed AD, ſo that it may be a mean proportional between FQ, FR, and through D draw BDC parallel to KO; BC will touch the hyperbola in D: for ſince the ſquare of AD is equal to the rectangle QFR, the point D is [cor. 17. 3.] in the hyperbola; and ſince KO, BC are parallel, and that KO is biſected in P by the ſtraight line PAD, BC is biſected in D; and therefore touches the hyperbola in the ſame point D.

PROP. XXV. THEOR.

If two ſtraight lines touch an hyperbola, or oppoſite hyperbolas, and cut the aſymptotes; the rectangle contained by the ſegments (of the aſymptotes) intercepted between the centre and the points where the one ſtraight line interſects the aſymptotes, is equal to the rectangle contained by the ſegments (of the aſymptotes)

totes) intercepted between the centre and the points where the other ſtraight line interſects the aſymptotes.

Let there be an hyperbola, with AB, AD for its aſymptotes, and let a ſtraight line BD touch it in C, and a ſtraight line GE touch the ſame, or the oppoſite hyperbola, in F; the rectangles BAD, EAG will be equal. Fig. 16.

From the points C, F draw CH, CK, and FL, FM parallel to the aſymptotes; then BCD touching the hyperbola, BC is equal to CD, and conſequently BA is the double of AH, and AD the double of HC; therefore the rectangle BAD is the quadruple of the rectangle CHA. It may be ſhewn in the ſame manner, that the rectangle EAG is the quadruple of the rectangle FMA: but, by propoſition 16. of this book, the rectangle CHA is equal to the rectangle FMA; the rectangle BAD is therefore equal to the rectangle EAG.

PROP.

PROP. XXVI. Theor.

If two ſtraight lines touching an hyperbola, or oppoſite hyperbolas, meet the aſymptotes; the ſtraight lines drawn between the points of concourſe are parallel to each other, and to the ſtraight line joining the points of contact.

Fig. 16. Let there be an hyperbola, with AB, AD for its aſymptotes; let BD touch it in C, and EG touch the ſame, or the oppoſite hyperbola, in F; BE, DG, when joined, are parallel to CF.

Since the rectangles BAD, EAG are equal, BA is to EA as GA to AD; therefore BE, GD are parallel: join DF, and let it meet BE in N; then ſince DF is to FN as GF to FE, that is, as [23. 3.] DC to CB, BN, CF are parallel.

Cor. Of two ſtraight lines touching an hyperbola, the ſegments between the aſymptotes are cut, proportionally, in the

point

point O where the two ſtraight lines interſect each other, and in C, F the points of contact.

PROP. XXVII. Theor.

Every ſtraight line drawn through the centre of an hyperbola, and paſſing within the angle adjacent to that within which the hyperbola is deſcribed, is a right diameter.

Let there be an hyperbola, AC, BC its aſymptotes, and draw any ſtraight line CE through the centre, and within the angle adjacent to the angle ACB; CE is a right diameter. Fig. 15. n. 2.

In BC produced take any point D, and through D to CE draw a ſtraight line DF parallel to the aſymptote CA; and having made DG equal to DC, join GF, and let GF meet CA in H: then, ſince GH meets the ſtraight lines CA, CD, which contain the angle adjacent to ACB, it muſt meet [20. 3.] the hyperbolas: let it meet them in the points K, L; KH, LG are, therefore,

fore, equal [14. 3.]: and becauſe CD, DG are equal, and CH, DF parallel, HF, FG are equal; the whole FK is, therefore, equal to the whole FL; and therefore CF is [4. def. 3.] a right diameter.

DEF. XI.

When a ſtraight line drawn through the centre of an hyperbola, and biſected in the centre, is parallel to a ſtraight line which touches the hyperbola, and equal to the ſegment (of the tangent) between the aſymptotes, it is named the *ſecond diameter* of the diameter drawn through the point of contact.

Cor. 1. Hence every ſecond diameter is a right diameter; for it paſſes within the angle adjacent to that within which the hyperbola is deſcribed.

Cor. 2. Hence a ſtraight line which joins the vertices of a tranſverſe diameter, and of its ſecond diameter, and thus cuts one of the aſymptotes, is parallel to the other aſymptote.

Fig. 15. n. 2. For let OCP be a tranſverſe diameter, and MCN its ſecond, and AOB a ſtraight line touching the hyperbola in the vertex of the tranſverſe OCP; MO, NO are [31. 1. Elem. and 11. def. 3.] parallel to CB, CA.

DEF. XII.

A third proportional to two diameters, one of which is a tranſverſe diameter, and the other the ſecond diameter of this tranſverſe, is named the *latus rectum*, or the *parameter* of that diameter which is the firſt of the three proportionals.

PROP. XXVIII. THEOR.

If from a point in an hyperbola to a tranſverſe diameter a ſtraight line be drawn parallel to the ſecond diameter of that tranſverſe, the ſquare of the tranſverſe diameter is to the ſquare of the ſecond diameter, as the rectangle contained by the ſegments of the tranſverſe which are between its vertices and the parallel,

lel, is to the ſquare of the parallel.

Fig. 17. n. 2. A*a* being a tranſverſe diameter, and B*b* its ſecond diameter, CG, CF the aſymptotes, and DE drawn parallel to B*b* from a point in the hyperbola to the tranſverſe A*a*; the ſquare of A*a* is to the ſquare of B*b* as the rectangle AE*a* is to the ſquare of DE.

Let DE meet the aſymptotes in F, G, and draw HAK touching the hyperbola in the vertex A; therefore, by def. 11. of this book, HA is equal and parallel to BC, and, of conſequence, parallel to FE; and, becauſe of the equiangular triangles, the ſquare of CE is to that of EF as the ſquare of CA to that of AH, that is, as the ſame ſquare of CA to the [4. cor. 23. 3.] rectangle FDG; the ſquare of CA is, therefore [19. 5. Elem.], to the ſquare of AH as the rectangle AE*a* to the ſquare of ED; the ſquare of A*a*, therefore, is to the ſquare of B*b* as the rectangle AE*a* to the ſquare of ED.

Cor. 1. The ſquares of ſtraight lines drawn from points of an hyperbola, or of the oppoſite hyperbola, to a tranſverſe diameter, and which are parallel to the ſecond diameter of this tranſverſe, are to one another, as the rectangles contained by the ſegments (of the tranſverſe diameter) intercepted between its vertices and thoſe parallels. See prop. 15. cor. 1. B. 2.

COR. 2. And, on the contrary, A being an hyperbola, having A*a* for a tranſverſe diameter, and B*b* for the ſecond diameter of A*a*; if from a point D to the tranſverſe A*a* a ſtraight line DE be drawn parallel to the ſecond diameter, and meeting the tranſverſe diameter produced in E, and the ſquare of CA be to the ſquare of CB as the rectangle AE*a* to the ſquare of ED; the point D is in the hyperbola. For ſince DE is parallel to BC, and conſequently to HK, which touches the hyperbola in the vertex of the tranſverſe diameter, it will neceſſarily meet the aſymptotes [3. cor. 18. 3.], and of conſequence the hyperbola: if, then, it does not meet the hyperbola in D, let it, if poſſible, meet it in another point *d*, on the ſame ſide of A*a* with the point D; therefore the rectangle AE*a* is to the ſquare of *d*E as the ſquare of CA to the ſquare of CB, that is, by hypotheſis, as

Fig. 17.

the rectangle AEa to the ſquare of DE; therefore dE, DE are equal; which is abſurd; DE, therefore, meets not the hyperbola in d, nor, as is evident, in any point but D.

COR. 3. Subſtitute the word *hyperbola* in place of *ellipſe*, and the third corollary of prop. 15. B. 2. becomes alſo a corollary from this propoſition.

PROP. XXIX. THEOR.

If from a point of an hyperbola to a ſecond diameter a ſtraight line be drawn parallel to the tranſverſe of that ſecond diameter; the ſquare of the ſecond diameter is to the ſquare of the tranſverſe diameter, as the ſum of the ſquares of half the ſecond diameter, and of the ſegment between the centre and the parallel, is to the ſquare of the parallel.

From D, a point of the hyperbola, to the ſecond diameter B*b*, draw DL parallel to the tranſverſe diameter A*a*; the ſquare of B*b* is to the ſquare of A*a* as the ſum of the ſquares of CB, CL is to the ſquare of DL. Fig. 17.

Through the point D draw DE parallel to BC; and ſince, by the preceding propoſition, the ſquare of CA is to the ſquare of CB as the rectangle AE*a* to the ſquare of ED; therefore, inverſely, and by prop. 12. 5. Elem. the ſquare of CB is to the ſquare of CA as the ſum of the ſquares of CB, ED to the ſquare of CA, together with the rectangle AE*a*, that is, as the ſum of the ſquares of CB, CL to the ſquare of EC or DL.

COR. 1. If from two points of an hyperbola, or of oppoſite hyperbolas, to a ſecond diameter, two ſtraight lines be drawn parallel to the tranſverſe of that ſecond diameter, the ſquare of the one ſtraight line is to the ſquare of the other, as the ſum of the ſquares of half the ſecond diameter, and of the diſtance between the one ſtraight line and the centre, to the ſum of the ſquares of half the ſame ſecond diameter,

meter, and of the diſtance between the other ſtraight line and the centre.

Cor. 2. And, on the other hand, if from a point D to a ſecond diameter BC of an hyperbola, a ſtraight line DL be drawn parallel to CA the tranſverſe of that ſecond diameter, and the ſquare of B*b* have the ſame ratio to the ſquare of A*a* that the ſum of the ſquares of CB, CL have to the ſquare of DL, the point D is in the hyperbola. For ſince the ſtraight line DL is parallel to the tranſverſe diameter AC, which is in the ſpace between the aſymptotes, it meets the ſtraight lines containing the angle adjacent to that within which the hyperbola is deſcribed, and, of conſequence, both the hyperbolas [20. 3.]; and, as in the ſecond corollary of the preceding propoſition, it may be proved, that it meets the hyperbola in D.

Cor. 3. With the proper alterations, the third corollary of the preceding holds as a corollary here.

PROP.

PROP. XXX. THEOR.

Let there be a tranſverſe diameter, and its ſecond diameter; any ſtraight line terminated both ways by the hyperbola, or the oppoſite hyperbolas, and parallel to either of theſe diameters, is biſected by the other of them; or, what is the ſame thing, a tranſverſe diameter, and its ſecond, are conjugate diameters.

COR. It is evident, that two diameters cannot be conjugated to the ſame diameter, whether it be a tranſverſe or a ſecond diameter.

PROP. XXXI. THEOR.

Let there be a tranſverſe diameter, and its ſecond diameter; any ſtraight line terminated both ways by an hyperbola, or oppoſite hyperbolas, and biſected by either of theſe diameters, is parallel to the other of them: and therefore ſtraight lines ordinately applied to either of theſe diameters are parallel to each other.

Cor. 1. There being a tranſverſe, and its ſecond diameter, ſtraight lines parallel to either of them, and which cut off equal ſegments of the other, between the points where they meet it and the centre, are equal; and if parallel to either diameter, and equal to each other, they cut off, between the centre and the points where they meet the other diameter, equal ſegments of this other diameter.

Fig. 17. n. 2. Theſe two propoſitions, and this firſt corollary, are demonſtrated from the 28th and 29th propoſitions, in the ſame manner in which the 9th and 10th propoſitions were demonſtrated from the 7th and 8th.

Cor. 2. If two or more parallels are terminated both ways by an hyperbola, or hyperbolas, the diameter which biſects the one, or one of them, biſects the other, or the reſt of them; for that parallel which is biſected, is parallel to the diameter conjugated to the biſecting diameter; the other, or the reſt, therefore, is, or are, parallel to the ſame conjugated diameter, and therefore is or are biſected by the other diameter [30. 3.].

COR. 3. A ſtraight line, on the other hand, which biſects two parallels terminated both ways by an hyperbola, or oppoſite hyperbolas, is a diameter: for if not, draw a diameter biſecting the one of the parallels; this diameter will biſect the other of them; but, by hypotheſis, there is another ſtraight line which biſects both; which is abſurd.

COR. 4. If a ſtraight line touch an hyperbola, a ſtraight line drawn through the point of contact, and which biſects a ſtraight line terminated both ways by the hyperbola, and parallel to the touching line, is a diameter; for a parallel to the tangent is [11. def. 3.] parallel to the diameter conjugated to that which paſſes through the point of contact: and if the ſtraight line drawn through the point of contact, and which biſects the ſtraight line mentioned, be not a diameter, draw a diameter through the point of contact; and this diameter will alſo biſect [30. 3.] the parallel to the touching line, or to the conjugated diameter; which is abſurd.

COR. 5. Two ſtraight lines terminated both ways by an hyperbola, or by oppoſite hyperbolas, and not paſſing through the centre, do not biſect each other: for if they are both terminated by the ſame hyperbola, or by oppoſite hyperbolas, draw a diameter through the point where they interſect each other; and then, by the propoſition, they will be both parallel to the diameter conjugated to this diameter; which is abſurd: but if the one of them be terminated by the hyperbola, and the other drawn between the oppoſite hyperbolas, it is evident that they cannot biſect each other.

PROP. XXXII. THEOR.

A ſtraight line drawn through the vertex of a tranſverſe diameter, and which is parallel to a ſtraight line ordinately applied to that diameter, touches the hyperbola; and if it touch the hyperbola, it is parallel to ſtraight lines ordinately applied to the tranſverſe diameter drawn through the point of contact.

Let there be an hyperbola, the aſymptotes of which are CF, CG; let A*a* be a tranſverſe diameter, and through the vertex A of A*a* draw HAK parallel to DM ordinately applied to the ſame A*a*; HK touches the hyperbola. Fig. 17.

The ſtraight line ordinately applied to the tranſverſe diameter A*a* is parallel [31. 3.] to the ſecond diameter conjugated to that tranſverſe; therefore HAK drawn through the vertex A of A*a*, is parallel to the ſame conjugated ſecond diameter; and therefore it touches the hyperbola [11. def. 3.]. And, on the other hand, if HK touch the hyperbola, it is parallel to the ſecond diameter [11. def. 3.] of CA: but the ordinate DM is parallel to the ſame [31. 3.]; therefore HK, DM are parallel to each other.

PROP. XXXIII. THEOR.

A ſtraight line that touches an hyperbola meeting a diameter, there being drawn from the point of contact a ſtraight line ordinately applied to that diameter; the ſemidiameter is a mean proportional between its ſegments intercepted between the centre and the ordinate, and between the centre and the tangent.

Caſe 1. When the touching line meets a tranſverſe diameter.

Fig. 18. There being an hyperbola, its aſymptotes AG, AH, a ſtraight line KCG being drawn touching that hyperbola in the point C, and meeting a tranſverſe diameter BAO in E, and CD drawn from the point of contact C, ſo as to be ordinately applied to BAO; AD, AB, AE are proportionals.

Through the vertex B let GBF be drawn

parallel to CD, and let it meet in the point N the tangent drawn through C, and let BM drawn parallel to HC meet AG in M, and let DC meet AH in L, and join KF, GH; then, becauſe GBF, drawn through the vertex of the diameter, is parallel to the ordinate DC, it [32. 3.] touches the hyperbola: and ſince the tangents HK, GF are cut proportionally in [cor. 26. 3.] the points C, B, N; CN is to NH as BN to NG; and therefore, becauſe of the parallels, LF is to FH as MK to KG; and ſince KF, GH are [26. 3.] parallel, FH is to FA as KG to KA; therefore, *ex æquo*, LF is to FA as MK to KA: and, by compoſition, LA is to FA as MA to KA; therefore, becauſe of the parallels, DA is to BA as BA to EA.

Caſe 2. When the tangent meets a ſecond diameter of a tranſverſe diameter.

The ſtraight line CE touching the hyperbola, and meeting both the ſecond diameter AB in E, and KAF (which is conjugated to AB) in G, and CD, CH being drawn from the point of contact C, ſo as to be ordinately applied to the conjugate diameters; AD, AB, AE are proportionals. Fig. 19.

For by the preceding caſe, AH, AF, AG are proportionals; therefore the ſquare of

AH is to the ſquare of AF as AH is [cor. 20. 6.] to AG; and, by diviſion, the rectangle KHF is to the ſquare of AF as HG to GA: but ſince CH is ordinately applied to AF, the rectangle KHF is to the ſquare of AF as the ſquare of CH or AD to the ſquare of AB; therefore the ſquare of AD is to the ſquare of AB as HG to GA, that is, as CH or AD to AE; and therefore AD, AB, AE are proportionals [converſ. cor. 20. 6. Elem.]

Fig. 18. Cor. 1. Since (in the caſe where the tangent and ordinate meet the tranſverſe BAO) AD, AB, AE are proportionals, DO is to DB as EO to EB, that is, the ſegments (of the diameter) between its vertices and the ordinate are to each other as the ſegments of the ſame between the tangent and the ſame vertices. The demonſtration is ſimilar to that given in the ſecond part of prop. 17. B. 2.

Cor. 2. When (in the ſecond caſe) the ordinate drawn from the point of contact paſſes through the extremity of the ſecond diameter, the tangent paſſes through the other extremity of the ſame diameter: for ſince the diſtance between the ordinate and the centre is equal to the half of the ſecond diameter, the diſtance between the tangent and the centre is equal to the ſame ſemidiameter.

PROP. XXXIV. THEOR.

If from a point C in an hyperbola a ſtraight line CD be ordinately applied to the diameter AB, and a ſtraight line CE be drawn from the ſame point; if the ſemidiameter be a mean proportional between the ſegments of AB, which are cut off towards the centre by theſe ſtraight lines; CE touches the hyperbola.

For if CE does not touch the hyperbola, let CP touch it; therefore, by the preceding propoſition, AD, AB, AP are proportionals: but, by the hypotheſis, AD, AB, AE are proportionals; which is abſurd: CE, therefore, touches the hyperbola. Fig. 18. 19.

PROP.

PROP. XXXV. THEOR.

If a ſtraight line touching an hyperbola meet a tranſverſe diameter, there being drawn from the point of contact a ſtraight line ordinately applied to the ſame diameter; the rectangle contained by the ſegments (of the diameter) intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to the rectangle contained by the ſegments between the ordinate and the vertices of the diameter; and the rectangle contained by the ſegments between the tangent and the centre, and between the tangent and the ordinate, is equal to the rectangle contained by the ſegments between the tangent and the vertices of the diameter. But if the tangent meet a ſecond diameter, there being drawn from the point of contact a ſtraight line ordinately applied to that ſecond diameter; the rectangle contained by the ſegments between the ordinate and the centre, and between the ordinate and the tangent, is equal to the ſum of the ſquares of the ſemidiameter and the ſegment between the ordinate and the centre; and the rectangle contained by the ſegments between the tangent and the centre, and between the tangent and the ordinate, is equal to the ſum of the ſquares of the ſemidiameter and the ſegment between the tangent and the centre.

Caſe 1. The ſtraight line which touches the hyperbola in C, meeting the tranſverſe Fig. 18.

diameter BAO in the point E, and an ordinate meeting the ſame diameter in D; the rectangle ADE is equal to the rectangle BDO, and the rectangle AED to BEO.

For ſince AD, AB, AE are proportionals, the rectangle DAE is equal to the ſquare of AB; and theſe equals being taken from the ſquare of AD, the remaining rectangle ADE, is equal to the remaining rectangle BDO [2. and 6. 2. Elem.]. Next, from the ſame equals, viz. the rectangle DAE and the ſquare of AB, take away the common ſquare [3. and 5. 2. Elem.] of AE, and the remaining rectangle AED is equal to the remaining rectangle BEO.

Fig. 19. Caſe 2. The tangent and ordinate drawn from the point C, meeting the ſecond diameter in the points E, D; ſince the rectangle EAD is equal to the ſquare of AB, add to each of theſe equals the ſquare of AD, and the rectangle ADE will be equal to [3. 2. Elem.] the ſum of the ſquares of AB, AD. Next, if to the ſame equals, to wit, the rectangle EAD and the ſquare of AB, the ſquare of AE be added; the rectangle AED will be equal to the ſum of the ſquares of AB, AE.

PROP.

PROP. XXXVI. THEOR.

If a ſtraight line touch an hyperbola it biſects the angle contained by the ſtraight lines drawn from the foci to the point of contact; and, on the contrary, if a ſtraight line biſect the angle contained by two ſtraight lines drawn from a point of an hyperbola to the foci, it touches the hyperbola.

There being an hyperbola, its tranſverſe axis AB, and the centre the point C, a ſtraight line DE touching it, and meeting the tranſverſe axis in E, and the ſtraight lines DF, DG being drawn from D the point of contact to the foci; the angles FDE, GDE are equal. Fig. 20.

From the point D draw DH perpendicular to the axis, and from the point A, which is the nearer (to D) of its vertices, place, in the axis produced, a ſtraight line AK equal to DF, and KB will [1. 3.] be equal to DG; and, by the fifth propoſition of this book, CK is to CH as CF to CA: but as CH to CA, ſo [33. 3.] is CA to CE; therefore, *ex æquali*, as CK to CA ſo is CF to CE; and, by converſion, as CK is to KA ſo is CF to FE; and by doubling the antecedents, twice CK is to KA as FG to FE: therefore, by diviſion, BK is to KA as GE to FE; and BK, KA are equal to DG, DF; therefore as DG to DF, ſo is GE to EF; and therefore the ſtraight line DE biſects the angle FDG [3. 6. Elem.].

If, on the contrary, a ſtraight line DE biſects the angle FDG, it touches the hyperbola: for if not, let another ſtraight line touch the hyperbola in the point D; this other ſtraight line will biſect the angle FDG, which, by the hypotheſis, is biſected by DE; which is abſurd. The demonſtration here might have been ſimilar to the ſecond demonſtration in prop. 11. B. 2.

PROP.

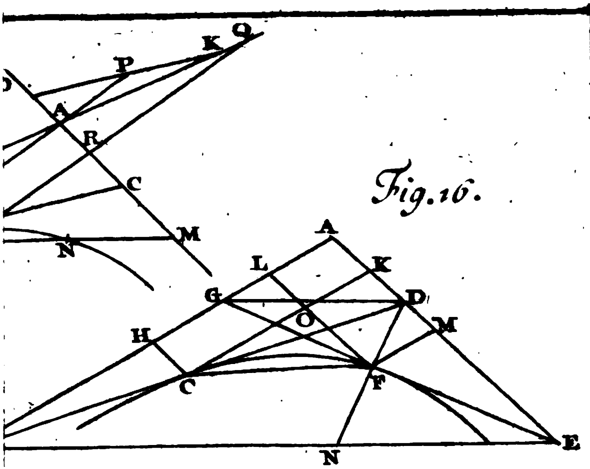


Fig. 16.

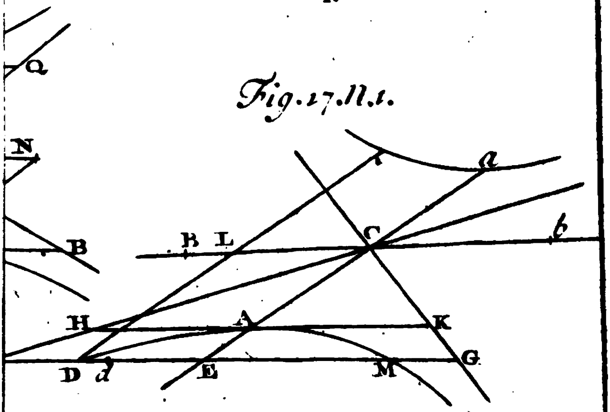


Fig. 17. N. 1.

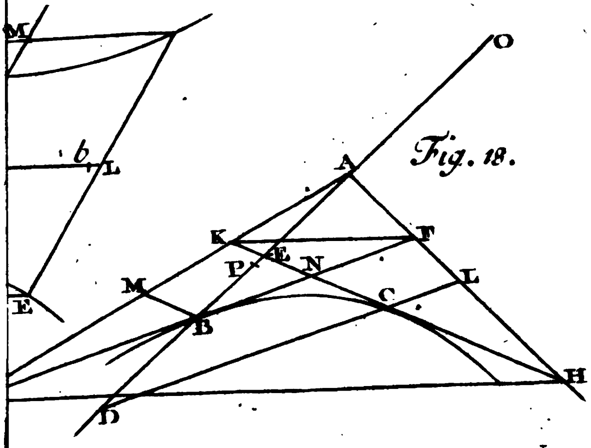


Fig. 18.

PROP. XXXVII. PROB.

Two ſtraight lines AB, CD which biſect each other at right angles in the point E, being given in poſition and magnitude; to deſcribe the oppoſite hyperbolas of which they may be the axes, and of which either of them AB may be the tranſverſe axis. Fig. 21.

Join AC, and from the point E place in AB produced two ſtraight lines EF, EG, each of them equal to AC; then, by means of a ſtring and of a ruler, the length of which exceeds that of the ſtring by a difference equal to AB, deſcribe with the foci F, G two oppoſite hyperbolas; theſe will paſs through the points A, B, and CD will be their ſecond axis.

For if the hyperbola paſſes not through A, let it paſs, if poſſible, through H; the exceſs, therefore, of HG above HF is equal to the exceſs of the length of the ruler above that of the ſtring, that is, by conſtruction, to the ſtraight line AB: but ſince BG is equal to AF, the exceſs of AG above AF is equal to the ſame AB; which is abſurd: the hyperbola, therefore, paſſes through A: and in like manner it may be ſhewn that it paſſes through B. Again, C, D are the extremities of the ſecond axis: for if C be not one of its extremities, let the point K, on the ſame ſide of the centre on which C is, be one of them; therefore KA being joined will be equal [6. def. 3.] to EF; and, by conſtruction, CA is equal to the ſame EF; therefore KA is equal to CA: which is abſurd.

D E F. XIII.

Fig. 22. If upon two ſtraight lines A*a*, B*b*, which biſect each other at right angles, two oppoſite hyperbolas AG, *ag* be deſcribed, and upon the ſame ſtraight lines other two oppoſite hyperbolas BK, *bk* be deſcribed, ſo that B*b*, the tranſverſe axis of the latter hyperbolas, may be the ſecond axis of the former, and that A*a*, the ſecond axis of the latter, may be the tranſverſe axis of the former; theſe four are named *conjugate hyperbolas*.

PROP.

PROP. XXXVIII. THEOR.

The aſymptotes of conjugate hyperbolas are common to the four hyperbolas.

Let there be conjugate hyperbolas the axes of which are A*a*, B*b*, and let the ſtraight lines CD, CE be the aſymptotes of two oppoſite hyperbolas, the tranſverſe axis of which is A*a*; the ſame ſtraight lines are the aſymptotes of the two oppoſite hyperbolas, the tranſverſe axis of which is B*b*. Fig. 22.

Through the vertex A draw DAE parallel to BC, and join DB, and produce it to F; therefore, by the 10th definition of this book, BC is equal and parallel to AD or AE; BD is therefore equal and parallel to CA: and becauſe the triangles FBC, CAE are equal and equiangular, BF is equal to CA; therefore CD, CF are [10. def.] aſymptotes of the hyperbola, the tranſverſe axis of which is B*b*, and the ſecond axis A*a*.

PROP.

PROP. XXXIX. THEOR.

If from a point G in one of four conjugate hyperbolas, a ſtraight line GH be drawn parallel to EC, one of the aſymptotes, and meeting the other in H, there being drawn from the point K in the adjacent hyperbola a ſtraight line KL parallel to either aſymptote, and meeting the remaining one in L; the rectangles GHC, KLC contained by the parallels, and the ſegments of the aſymptotes between the parallels and the centre, are equal. On the contrary, if the point G be in one of the conjugate hyperbolas, and the point K within the angle contained by the aſymptotes of the adjacent hyperbola, the rectangle KLC being at the ſame time equal to the rectangle GHC; the point K is in the adjacent hyperbola.

Let A*a*, B*b* be the axes of conjugate hyperbolas; join AB, and let it meet the aſymptote CD in M, and draw AD parallel to CB; then, ſince BC, AD are equal and parallel, the triangles CBM, ADM are ſimilar and equal; and conſequently AM, MB are equal: the rectangles AMC, BMC are therefore equal, and AB is parallel to the [2. cor. def. 10. 3.] aſymptote EC: therefore the rectangles GHC, KLC are equal to the rectangles AMC, BMC; and conſequently they are equal to each other. Fig. 24.

On the contrary, if G be a point in one of the conjugate hyperbolas, and the point K be within the angle DCF, and the rectangle KLC equal to the rectangle GHC, with the ſame conſtruction in other reſpects ſtill remaining; the point K is in the adjacent hyperbola: for ſince the rectangle KLC is equal to GHC, that is, to AMC, that is, to BMC, and that B is in the adjacent hyperbola; the point K is [4. cor. 18. 3.] in the ſame hyperbola.

Fig. 23. COR. 1. If a ſtraight line DE, intercepted between an hyperbola and one of the adjacent hyperbolas, is biſected by one of the aſymptotes, it is parallel to the other: for let DE meet the aſymptote CG in L, and to the other aſymptote draw DH, EK parallel to CG; then, ſince DL, LE are equal, and that DH, LC, EK are parallel, HC, CK are equal; and by the propoſition, the rectangles DHC, EKC are equal; DH, EK are therefore equal, and they are parallel; therefore DE, KH are parallel.

COR. 2. If DE be parallel to the aſymptote KH; DL, LE are equal: for by the propoſition, the rectangles DLC, ELC are equal.

COR. 3. If, laſtly, DE be parallel to the aſymptote HK, and DL, LE being equal, the point D be in one of the hyperbolas; the point E is in the adjacent hyperbola: for ſince DL, LE are equal, the rectangles DLC, ELC are equal; therefore, by the propoſition, E is in the adjacent hyperbola BE.

PROP.

PROP. XL. [*Prop.* 20. *B.* 2. *Apoll.*]

If a ſtraight line touch one of four conjugate hyperbolas, and through their centre two ſtraight lines be drawn, the one meeting the hyperbola in the point of contact, and the other parallel to the tangent, and meeting one of the adjacent hyperbolas; this other ſtraight line drawn parallel to the tangent, is the half of the ſecond diameter conjugated to the tranſverſe diameter drawn through the point of contact: and, on the contrary, if half a ſecond diameter be drawn conjugated to the tranſverſe diameter drawn through the point of contact, its extremity is in the adjacent hyperbola.

Let there be conjugate hyperbolas the aſymptotes of which are CG, CF, and let MF Fig. 23.

MF touch one of the hyperbolas in D; join CD, and draw CE parallel to MF, and meeting in the point E the hyperbola adjacent to that in which D is; CE is the half of a ſecond diameter conjugated to CD.

Through the points D, E draw the ſtraight lines DH, EK parallel to the aſymptote CG, and meeting the other aſymptote in H, K: then, ſince the ſtraight line MF touches the hyperbola in D, MD, DF are equal; therefore CH, HF are equal; therefore the rectangle DHF is equal to the rectangle DHC, that is, by the preceding propoſition, to the rectangle EKC; and ſince the triangles EKC, DHF are equiangular, EK is to DH as KC to HF; therefore the ſquare of EK is to the ſquare of DH, as the rectangle EKC to the rectangle DHF: but the rectangles EKC, DHF, as has been proved, are equal; therefore the ſquares of EK, DH are equal; and therefore EK, DH are likewiſe equal; conſequently EC, DF are [26. 1. Elem.] equal, and they are parallel; therefore CE is half [11. def. 3.] the ſecond diameter conjugated to CD.

If, on the contrary, CE (the ſame conſtruction remaining) be half the ſecond diameter conjugated to CD, its extremity E is in the hyperbola adjacent to that where the point D is: for CE is equal and [11. def. 3.] parallel to DF, and EK is parallel to DH; conſequently the triangles EKC, DHF are [26. 1. Elem.] equal; KC, conſequently, is equal to HF or HC, and EK to DH: and for this reaſon, the rectangle EKC is equal to the rectangle DHC, and the point D is in the hyperbola, and the point E is within the angle adjacent to that within which this hyperbola is deſcribed; the point E, therefore, is in the adjacent hyperbola (by part 2. of the preceding.]. Fig. 23.

COR. 1. If CD be half a tranſverſe diameter, and CE half the ſecond diameter conjugated to CD in the hyperbola AD; CE is a tranſverſe, and CD a ſecond diameter conjugated to CE in the adjacent hyperbola BE.

DE and ME being joined, let ME meet the other aſymptote CF in P; then, ſince DE is [2. cor. def. 11. 3.] parallel to CF, and that MF is biſected in D, MP is biſected in E, and the point E is in the adjacent hyperbola; therefore MP touches that hyperbola: and CD is equal and parallel to ME; for CE, MD are equal and parallel; therefore CD is the ſecond diameter conjugated to the tranſverſe diameter CE in the [11. def.] hyperbola BE.

Cor. 2. The ſame conſtruction ſtill remaining, the ſtraight line which joins the centre C, and the point of concourſe M of the tangents DM, EM, drawn through the vertices of the conjugate diameters, is an aſymptote: for if CM be not an aſymptote, let CQ, meeting DM in Q, be one; CE, therefore, is equal to DQ: but the ſame CE is equal to DM; becauſe DMEC is a [11. def.] parallelogram; DQ and DM are therefore equal: which is abſurd.

PROP. XLI. Theor.

If from the extremity of a ſecond diameter of an hyperbola a ſtraight line be drawn parallel to any tranſverſe diameter, and meeting the ſecond diameter conjugated to this tranſverſe; the

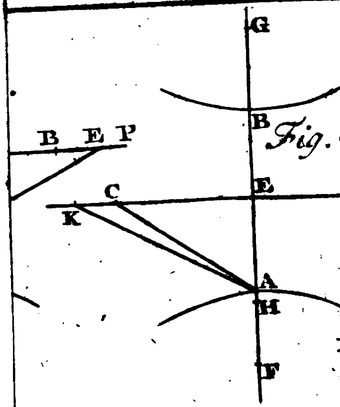


Fig. 21.

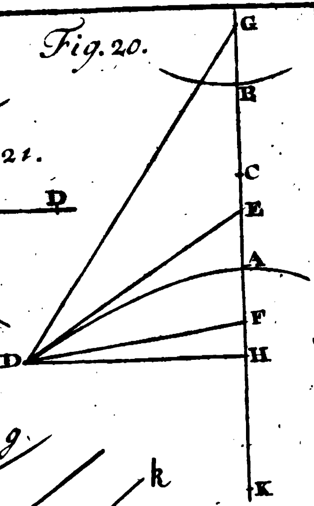
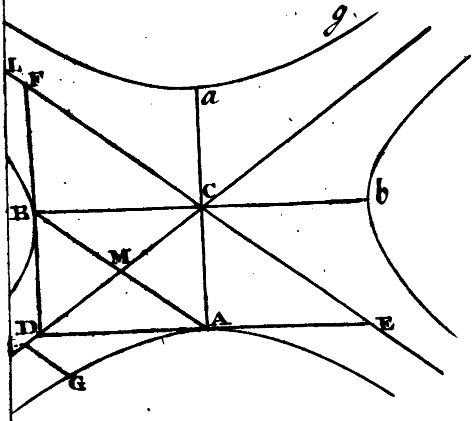
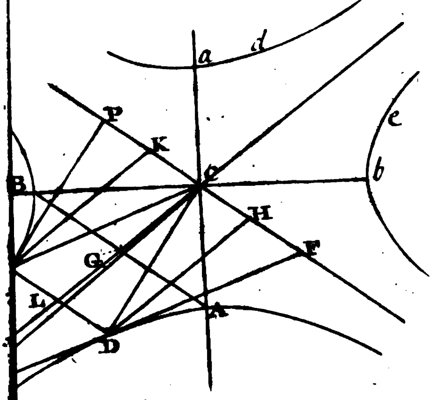


Fig. 20.





TAB.XII.

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the ſquare of the parallel is to the rectangle contained by the ſegments (of the diameter which the parallel meets) intercepted between the parallel and the vertices of the diameter met, as the ſquare of the tranſverſe diameter is to the ſquare of the ſecond diameter conjugated to it: And if from the extremity of a ſecond diameter a ſtraight line be drawn parallel to any other ſecond diameter, and meeting the tranſverſe diameter conjugated to this other ſecond diameter; the ſquare of the parallel is to the ſum of the ſquares of half the tranſverſe diameter and the ſegment (of the tranſverſe) intercepted between the centre and the parallel, as the ſquare of that other ſecond diameter is to the ſquare of the tranſverſe diameter conjugated to it.

Fig. 24. In the firſt caſe, let there be an hyperbola, the tranſverſe diameter of which is DC*d*, and let KC*k* be the ſecond diameter conjugated to DC*d*, and let CB be any other ſecond diameter, and from its extremity B draw a ſtraight line parallel to the tranſverſe diameter CD, and meeting in L the ſecond diameter conjugated to CD; the ſquare of BL is to the rectangle KL*k* as the ſquare of D*d* to the ſquare of K*k*.

For the points B, K, *k*, which are the extremities or vertices of the ſecond diameters, are in the adjacent hyperbolas; the tranſverſe diameter of which K*k* is conjugated to the ſecond [1. cor. 40. 3.] diameter D*d*; therefore the ſquare of BL is to the rectangle KL*k* as the ſquare of D*d* to the ſquare of K*k* [28. 3.].

As to the ſecond caſe, it is demonſtrated in the very ſame manner from the 29th propoſition of this book.

Cor. If from any point A of an hyperbola AD a ſtraight line AM be drawn ordinately applied to the right diameter K*k*, and from B, an extremity of any ſecond diameter CB to the ſame K*k*, a ſtraight line BL be drawn parallel to AM; the ſquare of BL, is to the ſquare of AM, as the rectangle KL*k* is to the ſum of the ſquares of the ſemidiameter KC and the ſegment CM between the centre and the ordinate. This is evident from the propoſition, and from the 29th of this book.

But if from any point A of an hyperbola AD a ſtraight line AE be drawn ordinately applied to the tranſverſe diameter D*d*, and from the extremity B of the ſecond diameter to the ſame D*d* a ſtraight line BF be drawn parallel to AE; the ſquare of BF is to the ſquare of AE, as the ſum of the ſquares of the ſemidiameter CD, and the ſegment CF, between the centre and BF, is to the rectangle DE*d*. This is demonſtrated from the propoſition, and from the 28th of this book.

PROP. XLII. THEOR.

Fig. 24. If from the extremity B of a ſecond diameter CB a ſtraight line BF be drawn parallel to ſtraight lines ordinately applied to any diameter D*d*, and BH be drawn parallel to the diameter CA, which is conjugated to CB, and meeting the diameter D*d* in H; the ſemidiameter CD is a mean proportional between CF and CH.

For the extremity B of the ſecond diameter is in the adjacent hyperbola, and CA is a [1. cor. 40. 3.] ſecond diameter of that adjacent hyperbola; therefore BH touches the ſame [11. def.]; and therefore CF, CD, CH are [33. 3.] proportionals.

COR. The 35th propoſition is equally true when accommodated to this caſe.

PROP.

PROP. XLIII. THEOR.

If from the extremities of two conjugate diameters of an hyperbola ſtraight lines be drawn ordinately applied to any third tranſverſe diameter; the ſquare of the ſegment (of the third diameter) intercepted between the centre and the ordinate drawn from the extremity of the tranſverſe diameter, is equal to the ſquare of the ſegment of the ſame third diameter between the centre and the ordinate drawn from the extremity of the other of the conjugate diameters, together with the ſquare of half the third diameter; and the ſquare of the ſegment (of the third diameter) intercepted between the centre and the ordinate drawn from the extremity ty of the ſecond diameter, is equal to the rectangle contained by the ſegments (of the third diameter) between the ordinate drawn from the extremity of the other of the conjugate diameters and the vertices of the ſame third diameter.

Fig. 34. Let there be oppoſite hyperbolas, in which CA is the half of a tranſverſe diameter, and let CB be the ſecond diameter conjugate to CA, and let D*d* be any other tranſverſe diameter, and from the extremities A, B draw AE, BF ordinately applied to D*d*; the ſquare of EC is equal to the ſquare of FC together with the ſquare of CD; and the ſquare of FC is equal to the rectangle DE*d*.

To the diameter D*d* draw AG parallel to BC, and BH parallel to AC; therefore, becauſe of the parallels, the triangles CAG, HBC are equiangular; and ſince AE, BF are parallel, CAE, HBF are alſo equiangular; CE, conſequently, is to HF as CA to BH, that is, as CG to CH: and ſince

CD is a mean proportional both between CE and CG, and between CF and CH [33. and 42. 3.], CF is to CE as CG to CH; and therefore CF is to CE as CE to HF; conſequently the ſquare of CE is equal to the rectangle CFH: but the rectangle CFH is equal to the ſum of the ſquares of CF, CD; conſequently the ſquare of CE is equal to the ſum of the ſame ſquares of CF, CD: take the ſquare of CD from each of theſe equals, and there will remain the rectangle DE*d* equal to the ſquare of CF.

COR. Hence the ſemidiameter CD, to which the ordinates are drawn, is to the conjugate ſemidiameter CK, as the diſtance between the one ordinate and the centre is to the remaining ordinate: for the ſquare of CD is to the ſquare of CK as the rectangle DE*d* is to the ſquare of EA, that is, according to the propoſition, as the ſquare of CF is to the ſquare of EA; and therefore CD is to CK as CF to EA. Again, becauſe the ſquare of CD is to the ſquare of CK as the ſum of the ſquares of CF, CD to the ſquare of BF, that is, by the propoſition, as the ſquare of CE to the ſquare of BF; CD is to CK as CE to BF.

PROP.

PROP. XLIV. THEOR.

The exceſs of the ſquares of any conjugate ſemidiameters is equal to the exceſs of the ſquares of the halves of the axes, if the conjugate diameters be unequal: and if any one diameter be equal to its conjugate, any other diameter is alſo equal to its conjugate; and in this caſe, the angle contained by the aſymptotes is a right angle.

Fig. 24. Let CA, CB be conjugate ſemidiameters, and CD, CK the halves of the axes, and from A, B draw the ſtraight lines AE, AM and BF, BL ordinates to the axes; therefore the exceſs of the ſquares of CA, CB is equal to the exceſs, by which the ſum of the ſquares of CE, EA differs from the ſum of the ſquares of CL, LB: but, by the preceding, the ſquare of CE is equal to the ſum of the ſquares of CF, CD; and by the ſame propoſition, the ſquare of CL is equal

to the ſum of the ſquares of CM, CK; therefore the exceſs of the ſquares of CA, CB is equal to the exceſs by which the ſum of the ſquares of CF, CD, EA differs from the ſum of the ſquares of CM, CK, LB; and the ſquares of CF, LB are equal; as alſo the ſquares of EA, CM: therefore, if theſe equals be taken away, the exceſs by which the ſum of the ſquares of CF, CD, EA differs from the ſum of the ſquares of CM, CK, LB, is equal to the exceſs by which the ſquare of CD differs from the ſquare of CK; and therefore the exceſs of the ſquares of CA, CB is equal to this ſame exceſs.

Otherwiſe: Let AB, AC be the halves of any two tranſverſe diameters in an hyperbola, AD, AE the aſymptotes; and draw the ſtraight lines BD, CE touching it in the points B, C, and meeting the aſymptotes in D, E; therefore, by the 11th def. and prop. 30. of this book, BD is equal to half the ſecond diameter conjugated to AB; and CE, in like manner, is equal to half the ſecond diameter conjugate to AC: it is to be proved, that the exceſs of the ſquares of AB, BD is equal to the exceſs of the ſquares of AC, CE. Fig. 25.

Through the points B, C draw BF, CH parallel to the aſymptotes, and BG, CK perpendicular to them: therefore the rectangles AFB, AHC are equal [1. cor. 16. 3.]; and, of conſequence, AF is to AH as HC to FB, that is, ſince the triangles are equiangular, as HK to FG; conſequently the rectangles AFG, AHK are equal, and their quadruples are equal: and ſince, through the point of contact B, a ſtraight line BF is drawn parallel to the aſymptote AE; DF, FA are equal; conſequently DG is equal to AF, together with FG: and hence four [8. 2. Elem.] times the rectangle AFG is equal to the exceſs of the ſquares of DG, GA, that is, ſince the triangles DGB, AGB are right-angled, to the exceſs of the ſquares of DB, BA. It may in the ſame manner be ſhewn, that four times the rectangle AHK is equal to the exceſs of the ſquares of EC, CA; and four times the rectangle AFG, as hath been proved, is equal to four times the rectangle AHK; conſequently the exceſs of the ſquares of DB, BA is equal to the exceſs of the ſquares of EC, CA.

But if in an hyperbola any tranſverſe diameter AB is equal to the ſecond diameter BD conjugated to it, any other tranſverſe diameter in the ſame hyperbola is alſo equal to its conjugate ſecond diameter, and the angle contained by the aſymptotes is a right angle: for ſince DB, BA, and DF, FA are equal, and BF common, in the triangles DBF, ABF; the angle BFD, and of conſequence the angle EAD, is a right angle; and ſince EAD is a right angle, the angle CHE is alſo a right angle; and EH, HA are equal, and CH common; therefore EC, CA are equal.

PROP. XLV. THEOR.

If through the vertices of two conjugate diameters four ſtraight lines be drawn touching conjugate hyperbolas, the parallelogram formed by them is equal to that formed by the tangents drawn through the vertices of any other two conjugate diameters.

Let AB, CD be conjugate diameters, and Fig. 26.

through their vertices draw tangents, meeting each other in K, L, M, N; and let EF, GH be any other conjugate diameters, and through the vertices of theſe draw tangents, meeting each other in O, P, Q, R; the figures KLMN, OPQR are parallelograms, and equal.

Let S be the centre of the hyperbola; and ſince both KN, LM, and KL, MN are [3. cor. 23. 3.] parallel, the figure KLMN is a parallelogram. For a like reaſon, OPQR is a parallelogram: and ſince AK, CK touch the hyperbolas in the vertices of conjugate diameters, the point K where they meet is in an aſymptote; and the reſt of the angles, as may be proved in like manner, of the parallelograms, have their vertices in the aſymptotes: therefore the aſymptotes are the diagonals of the parallelograms; conſequently the parallelogram KLMN is the quadruple of the triangle KSN, and the parallelogram OPQR the quadruple of the triangle OSR: but the triangles KSN, OSR are equal, becauſe the rectangles KSN, OSR are equal [25th of this book, and 15. 6. Elem.]; therefore the parallelograms KLMN, OPQR are alſo equal.

qual. This propoſition might alſo have been demonſtrated like prop. 20. B. 2.

PROP. XLVI. THEOR.

If two conjugate diameters of an hyperbola meet a ſtraight line touching the hyperbola, the rectangle contained by the ſegments (of the tangent) intercepted between the point of contact and the conjugate diameters, is equal to the ſquare of the ſemidiameter conjugated to that diameter which paſſes through the point of contact.

Let ACB, DCE be two conjugate diameters, and let a ſtraight line which touches the hyperbola in F meet them in the points G, H, and let CK be the ſemidiameter conjugated to CF; the rectangle GFH is equal to the ſquare of CK. Fig. 274

From the points F, K draw to AB the ſtraight lines FM, KL parallel to DE: then, becauſe of the parallels, GM is to MC as

GF to FH; conſequently the rectangles GMC, GFH are ſimilar: and becauſe the triangles GMF, CLK are equiangular, GM is to GF as CL to CK; therefore the rectangles GMC, GFH, and the ſquares of CL, CK, which are four ſimilar rectilineal figures, are proportionals, being ſimilarly deſcribed upon the four proportional ſtraight lines GM, GF, CL, CK: but the rectangle GMC is equal to [35. 3.] the rectangle AMB, that is, to the [43. 3.] ſquare of CL; therefore the rectangle GFH is equal to the ſquare of CK.

PROP. XLVII. THEOR.

If from a point of an hyperbola a ſtraight line be drawn ordinately applied to a tranſverſe diameter, the rectangle contained by the ſegments (of the diameter) intercepted between its vertices and the ordinate, is to the ſquare of the ſegment (of the ordinate) intercepted between the hyperbola and the diameter, as

Fig. 24.

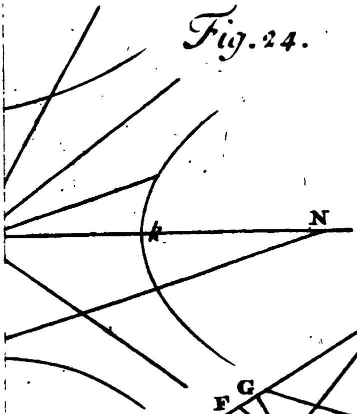
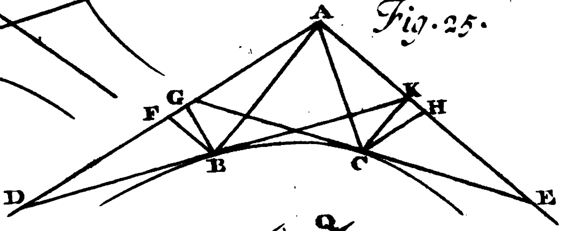
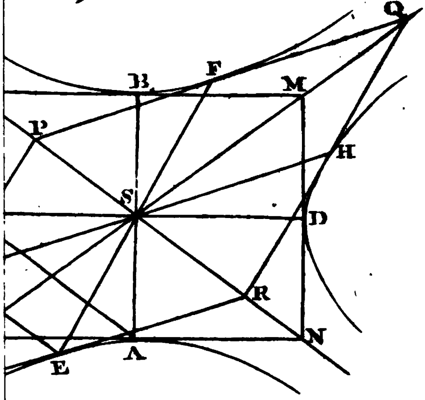
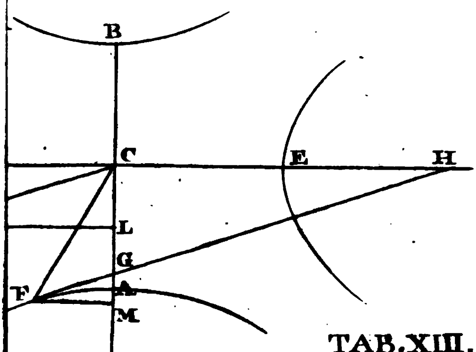


Fig. 25.







TAB. XIII.

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as the diameter is to its *latus rectum*; and if a ſtraight line be drawn ordinately applied to the ſecond diameter of the tranſverſe, the ſum of the ſquares of half the ſecond diameter, and of its ſegment between the ordinate and the centre, is to the ſquare of the ordinate, as the ſecond diameter is to its *latus rectum.*

There being a tranſverſe diameter AB, DE the ſecond diameter conjugated to it, AH the *latus rectum* of AB, and FG drawn from the point F in the hyperbola, ſo as to be ordinately applied to AB; the rectangle AGB is to the ſquare of FG as AB to AH: and FK being drawn ordinately applied to DE; the ſum of the ſquares of CD, CK is to the ſquare of FK as DE to its *latus rectum* L. Fig. 28.

Caſe 1. Since AB, DE, AH are proportionals, AB is to AH as the ſquare of AB to the ſquare of DE, that is, as the rectangle AGB to the ſquare of FG.

Caſe 2. And ſince DE, AB, L are proportionals, DE is to L as the ſquare of DE is to the ſquare of AB, that is, as the ſum of the ſquares of CD, CK is to the ſquare of KF.

PROP. XLVIII. Theor.

Fig. 28. If from a point F in an hyperbola a ſtraight line FG be ordinately applied to a tranſverſe diameter AB, and from the vertex of that diameter a ſtraight line AH be drawn perpendicular to AB, and equal to its *latus rectum*; the ſquare of the ordinate is equal to the rectangle applied to the *latus rectum*; which rectangle has for its breadth the abſciſſa between the ordinate and the vertex, and exceeds by a figure ſimilar, and ſimilarly ſituated, to that which is contained by the diameter and the *latus rectum*.

BH being joined, GM drawn from the point G parallel to AH, and meeting BH in M, and MN drawn through the point M parallel to AB, and meeting AH in N; the rectangles MNHO, BAHP are completed.

Then ſince the rectangle AGB is to the ſquare of FG, as AB to AH, that is, as Gβ to GM, that is, as the rectangle AGB to the rectangle AGM; AGB is to the ſquare of GF as the ſame AGB to the rectangle AGM; therefore the ſquare of GF is equal to the rectangle AGM; which having the abſciſſa AG for its breadth, is applied to the *latus rectum* AH, and exceeds the rectangle HAGO by the rectangle MNHO ſimilar to BAHP. From that ſquare's being equal to the exceeding rectangle, Apollonius named the curve line which is the ſubject of this third book, the *hyperbola*.

COR. It is evident, that the ſquare of GF would be equal to the rectangle AGM, though AH were not at right angles to AB.

PROP.

PROP. XLIX. PROB.

Fig. 29. **A ſtraight line AB being given in poſition and magnitude, and a point F being given; to deſcribe an hyperbola, of which AB may be the tranſverſe axis, and which may paſs through the point F; but the given point muſt be ſo ſituated, that a perpendicular drawn from it towards AB may fall upon AB produced.**

Draw FG at right angles to AB, and find a ſtraight line DE ſuch, that the ſquare of AB may be to that of DE as the rectangle AGB to the ſquare of FG: let AB, DE biſect each other at right angles; then, with AB, DE as axes, and making the former the tranſverſe axis, deſcribe an [37. 3.] hyperbola AF; this hyperbola will paſs [2. cor. 28. 3.] through the point F.

PROP. L. PROB.

A ſtraight line DE being given in poſition and magnitude, and a point F being given; to deſcribe an hyperbola, of which DE may be the ſecond axis, and which may paſs through F. Fig. 29.

Biſect DE in C, and draw FH perpendicular to DE, and find a ſtraight line AB ſuch, that the ſquare of DE may be to that of AB as the ſum of the ſquares of CD, CH to the ſquare of FH; let DE, AB biſect each other at right angles; then with the axes AB, DE, and making the former the tranſverſe axis, deſcribe [37. 3.] an hyperbola; this hyperbola will paſs through the point F [2. cor. 29. 3.].

PROP. LI.

Of all tranſverſe diameters in an hyperbola the tranſverſe axis is the leaſt; and the angle contained by any other tranſverſe diameter, and a tangent drawn through its vertex, is leſs than a right angle.

Fig. 29. Let there be an hyperbola, CA the half of its tranſverſe axis, and CF the half of any other diameter; from F, the vertex of CF, draw FG perpendicular to the axis CA; therefore CF is greater than CG; and conſequently much greater than CA: Draw a ſtraight line touching the hyperbola in the point F, and meeting the axis CA in K; and ſince the angle CFG is acute, CFK muſt be ſtill more acute.

PROP. LII. PROB.

Of an hyperbola AF given in poſition, to find a diameter, the centre, the axes, the aſymptotes, and the foci.

Draw two parallel ſtraight lines, and let them be terminated both ways by the hyperbola; and the ſtraight line which biſects them is a [3. cor. 31. 3.] diameter; and any other diameter may be found in the ſame manner; and the point where two diameters thus found meet each other is the [4. def. 3.] centre. But if two oppoſite hyperbolas be given in poſition, the point which biſects the diameter firſt found is the centre.

Take in the hyperbola any point F, and from the centre C draw CF, and from the centre C, with the diſtance CF, deſcribe a circle; if this circle meet the hyperbola no where but in the point F, CF is the leaſt of the tranſverſe diameters, and is, conſequently, the tranſverſe axis: but if the circle meet the hyperbola again in another point L, join FL, and let it be biſected Fig. 29.

ſected in the point G; join alſo CG, and let it meet the hyperbola in A; CA is half the tranſverſe axis: For ſince FG, GL are equal, FL is ordinately applied to the diameter CG; and conſequently a ſtraight line which is drawn through the vertex A parallel to FL touches the hyperbola [32. 3.]; and the angle contained by this tangent, and the diameter CA, is a right angle, for the angle FGA is a right angle; therefore, by the preceding propoſition, CA is the tranſverſe axis.

Next, in order to find the ſecond axis, draw through the centre C a ſtraight line at right angles to CA, and in that ſtraight line take CD, and let the ſquare of CA have the ſame ratio to that of CD which the rectangle BGA has (CB being made equal to CA) to the ſquare of GF; CD will be the ſecond axis, as is evident from prop. 7. of this book.

Laſtly, having found the axes, find the aſymptotes from def. 10.

But if two oppoſite hyperbolas be given in poſition, the aſymptotes may be found more eaſily in this manner. Draw through the centre C any tranſverſe diameter AB; draw likewiſe a ſtraight line parallel to AB, and terminated in the hyperbolas in the points O, P; and to the ſtraight line OP apply, on both ſides, a rectangle equal to the ſquare of CA, and deficient by a ſquare; which is poſſible, ſince CA is leſs than the half of OP [4. cor. 15. 3.]; and let Q, R be the points of application; CQ, CR, when joined, will be the aſymptotes [3 cor. 15. 3.]. The foci are found in the manner mentioned in the 57th propoſition,

PROP. LIII. PROB.

The aſymptotes of an hyperbola being given in poſition, and a point in it being given; to find the axes of the hyperbola, and to deſcribe it.

Let AC, BC be the aſymptotes given in poſition, and D be the given point. Suppoſe the problem ſolved, to wit, let FCE, GCH be the axes, the former of which, as it is within the angle ACB, in which the point D is, muſt be the tranſverſe axis: Draw through E a ſtraight line parallel Fig. 30.

rallel to GH, and let this parallel meet the aſymptotes in the points K, L; conſequently KL is equal [10. def. 3.] to GH, and is biſected in E: and ſince in the triangles KEC, LEC, KE, EL are equal, and the angles at E right angles; the angles ECK, ECL are equal: but the angle KCL is given; conſequently its half KCE is given: and KC, and the point C, are given in poſition; conſequently CE is given [29. dat.] in poſition: through the given point D let DMN be drawn parallel to CE, and let it meet the aſymptotes in M, N; DN is therefore given in [28. dat.] poſition, and the points M, N [25. dat.]; therefore DM, DN are given in magnitude [26. dat.]; the rectangle MDN is conſequently given in magnitude: but the ſquare of CE is equal to this rectangle [1. cor. 15. 3.]; the ſquare of CE is, therefore, given in magnitude; and conſequently CE is given [55. dat.] in magnitude; but, as hath been proved, it is alſo given in poſition; the point E, therefore, and the ſtraight line KEL, are [27. 29. dat.] given in poſition; and conſequently KEL is given in magnitude, becauſe CA, CB are given in poſition: and GH is parallel and equal to KL; conſequently GH is given in magnitude; but it is alſo given in poſition, becauſe C is given, which biſects it; the axes, therefore, EF, GH are given in poſition and magnitude; therefore the hyperbola may be deſcribed by prop. 37. of this book.

The compoſition is as follows. Let the angle ACB be biſected by the ſtraight line CE; and having drawn DMN parallel to CE, make the ſquares of CE, CF each of them equal to the rectangle MDN; and through E draw KEL perpendicular to CE, and meeting the ſtraight lines AC, CB in the points K, L; and through C let GCH be drawn equal and parallel to KEL, ſo that at the ſame time it may be biſected in C: then, with the axes EF, GH, and making the former of them the tranſverſe axis, deſcribe an hyperbola; AC, BC will be its aſymptotes, and it will paſs through the point D: for ſince, by conſtruction, KEL is equal and parallel to the ſecond axis GH, and is biſected in E; CK, CL are [def. 10. 3.] the aſymptotes, and the rectangle MDN is equal to the ſquare of CE; the point D, therefore, is in the hyperbola.

And the aſymptotes AC, BC being given, and a point D of an hyperbola, as many points of that hyperbola, or of the oppoſite hyperbola, as may be thought neceſſary, may be found, by drawing through D any number of ſtraight lines ADB, D*ab*, meeting the aſymptotes in A, B, and *a*, *b*; and taking BO, *bo* equal to AD, *a*D, in ſuch a manner, that the two points D, O, and the two D, *o*, may be either both within or both without the points A, B, and *a*, *b*; for the points O, *o* will be [19. 3.] in the hyperbolas.

PROP. LIV. PROB.

Fig. 31. Two ſtraight lines ACB, DCE, which biſect each other in C, being given in poſition and magnitude; to deſcribe two oppoſite hyperbolas, which may have AB for a tranſverſe diameter, and DE for the ſecond diameter conjugated to AB.

Suppoſe what is required done, and let CF, CG be the aſymptotes; through A, the vertex of the tranſverſe diameter, draw a ſtraight line parallel to DE, and let it meet the aſymptotes in F, G; FG, therefore, is [11. def. 3.] equal to DE, and is biſected in A: but DE is given in magnitude; therefore FG is given in magnitude; and of conſequence its half AF is alſo given in magnitude: but AF is given [28. dat.] in poſition, ſince it is drawn through a given point A parallel to DE given in poſition; conſequently the point F is [27. dat.] given: In like manner the point G is given; and the point C is given; therefore the aſymptotes CF, CG are given in poſition, and the point A is given in the hyperbola; which may, therefore, be deſcribed by the preceding propoſition.

The compoſition is as follows. Through the vertex A of the tranſverſe diameter draw a ſtraight line FAG equal and parallel to the ſecond diameter DE, and in ſuch a manner that it may be biſected in A; join CF, CG, and, by the preceding propoſition, deſcribe an hyperbola, which may have for its aſymptotes the ſtraight lines CF, CG, and which may paſs through the point A: and, in the ſame manner, by employing the point B, deſcribe the oppoſite hyperbola; AB will be a tranſverſe diameter in theſe hyperbolas, and DE the ſecond diameter conjugated to it.

For ſince CF, CG are the aſymptotes of the hyperbolas, and that through the point A, in the one of the hyperbolas, there is drawn a ſtraight line FAG, which is biſected in A; FG touches [23. 3.] the hyperbola in A: and DE is equal and parallel to FG, and is biſected in the centre C; therefore DE is the ſecond diameter conjugated [11. def. and 30. 3.] to the tranſverſe diameter AB.

PROP. LV. PROB.

The poſition and magnitude of a diameter of an hyperbola being given, and the poſition being given of a ſtraight line which is ordinately applied to that diameter from a given point of the hyperbola; to deſcribe the hyperbola.

Caſe 1. When the given diameter is a tranſverſe diameter of the hyperbola.

Let AB be the given tranſverſe diameter, to which a ſtraight line HK, given in poſition, is ordinately applied from a given point H of the hyperbola; and let AB be biſected in C, and through C draw a ſtraight line parallel to HK; and in that ſtraight line take CD and CE equal to each other, ſo that the rectangle AKB may be to the ſquare of HK as the ſquare of AC to the ſquare of CD or CE; and, by the preceding propoſition, deſcribe two oppoſite hyperbolas, of which AB may be a tranſverſe diameter, and DE the ſecond diameter conjugated to AB; one of theſe hyperbolas will paſs through the point H [2. cor. 28. of this book.]. Fig. 31.

Caſe 2. When the given diameter is a ſecond diameter.

Let DE be the given ſecond diameter, to which HL given in poſition, and drawn from H, a given point in the hyperbola, is ordinately applied; and let DE be biſected in C, and through C draw a ſtraight line parallel to HL, and in that ſtraight line take CA and CB equal to each other, ſo that the ſum of the ſquares of LC, DC may be to the ſquare of HL as the ſquare of DC to the ſquare of AC or CB; and, by the preceding propoſition, deſcribe two oppoſite hyperbolas, having AB for a tranſverſe diameter, and CD for the ſecond diameter conjugate to AB: of theſe hyperbolas, the one which lies on the ſame ſide of DE with the point H, will paſs through the point H [2. cor. 29. 3.].

PROP. LVI. THEOR.

If a cone cut by a plane through the axis, be cut likewiſe by a ſecond plane, cutting its baſe in the direction of a ſtraight line perpendicular to the baſe of the triangle through the axis; and the common ſection of the triangle through the axis, and the ſecond plane, meet, without the vertex of the cone, a ſide of the triangle through the axis; the line which is the common ſection of the ſecond plane, and the conical ſurface, is an hyperbola, having for a tranſverſe diameter the common ſection of the triangle through the axis and the ſecond plane.

Let there be a cone, its vertex the point A, and baſe the circle BC; let it be cut by a plane through the axis, and let the triangle ABC be the ſection; let it be cut alſo by another plane, cutting its baſe in the direction of the ſtraight line DE perpendicular to BC, the baſe of the triangle ABC; and let the ſection made in the ſurface of the cone be the line DFE; and let the ſtraight line FG, the common ſection of the triangle through the axis, and the ſecond plane, be produced, and without the vertex of the cone, and in the point H, let it meet the ſide AC of the triangle ABC; the line DFE is an hyperbola, which has FG for one of its tranſverſe diameters. Fig. 32.

For in the ſection DFE take any point K, and through K to FG draw KL parallel to DE, and through L draw MN parallel to BC; the plane, therefore, which paſſes through KL, MN is [15. 11. Elem.]. parallel to the plane through DE, BC, that is, to the baſe of the cone; and therefore the [23. 1.] plane through KL, MN is a circle, of which MN is a diameter: but KL is [10. 11. Elem.] perpendicular to MN, becauſe DE is perpendicular to BC; the rectangle, therefore, MLN is equal to the [35. 3. Elem.] ſquare of KL: and in like manner, the rectangle BGC is equal to the ſquare of DG; the ſquare of DG is therefore to the ſquare of KL as the rectangle BGC to the rectangle MLN: but BG is to ML as FG to FL; and GC is to LN as GH to LH; therefore the ratios compounded of theſe ratios are the ſame to one another; and therefore the rectangle BGC is to [23. 6. Elem.] the rectangle MLN as the rectangle FGH to the rectangle FLH: hence, in like manner, the ſquare of DG is to the ſquare of KL as the rectangle FGH to the rectangle FLH: Deſcribe, therefore, an hyperbola [55. 3.], of which FH may be a tranſverſe diameter, and in which DG may be ordinately applied to FH: and

ſince, by conſtruction, the point D is in this hyperbola, the point K is likewiſe in it [3. cor. 28. 3.]. And the ſame thing may be proved with regard to all the points of the ſection DEF.

THE END.